Section 11.4: The Comparison Tests

The Direct Comparison Test Suppose that Σa_n and Σb_n are series such that

$$b_n \geqslant a_n \geqslant 0 \quad \text{for all } n.$$

(i) If Σb_n is convergent then Σa_n is also convergent.

(ii) If Σa_n is divergent then Σb_n is also divergent.

One series we tend to use to compare other series (not always, though) is the <u>p-series</u>, $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where p is any real number. ***You will be expected to remember the following crucial fact about the convergence and the divergence of p-series.***

$\boxed{1}$ The p-series $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p}$ is convergent if $p > 1$ and divergent if $p \leqslant 1$.

Another series that can be used for comparison is the <u>geometric series</u>, $\sum_{n=1}^{\infty} ar^{n-1}$, where $a \neq 0$. Remember that the geometric series converges to $\frac{a}{1-r}$ if $|r| < 1$ and it diverges if $|r| \geq 1$.

Problem 1. Use the Direct Comparison Test to determine whether the following series are convergent or divergent. **You do not have to determine the sum if the series is convergent.**

(a) $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt[3]{n}}{\sqrt{n^3+4n+3}}$ (b) $\displaystyle\sum_{k=1}^{\infty} \frac{k\,\sin^2(k)}{1+k^3}$ (**Hint:** $\sin^2(k) \leq 1$).

(c) $\displaystyle\sum_{n=1}^{\infty} \frac{4^{n+1}}{3^n-2}$ (**Hint:** Compare to a geometric series.) (d) $\displaystyle\sum_{k=1}^{\infty} \frac{(2k-1)(k^2-1)}{(k+1)(k^2+4)^2}$

The Limit Comparison Test Suppose that $\sum a_n$ and $\sum b_n$ are series with positive terms. If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c$$

where c is a finite number and $c > 0$, then either both series converge or both diverge.

PROOF Let m and M be positive numbers such that $m < c < M$. Because a_n/b_n is close to c for large n, there is an integer N such that

$$m < \frac{a_n}{b_n} < M \qquad \text{when } n > N$$

and so

$$mb_n < a_n < Mb_n \qquad \text{when } n > N$$

If $\sum b_n$ converges, so does $\sum Mb_n$. Thus $\sum a_n$ converges by part (i) of the Direct Comparison Test. If $\sum b_n$ diverges, so does $\sum mb_n$ and part (ii) of the Direct Comparison Test shows that $\sum a_n$ diverges. ■

Problem 2. Use the Limit Comparison Test to determine whether the following series are convergent or divergent AND investigate why the Direct Comparison Test is not useful for each of the series. **You do not have to determine the sum if the series is convergent.**

(a) $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\sqrt{n^2-1}}$ (b) $\displaystyle\sum_{n=1}^{\infty} \frac{5+2n}{(1+n^2)^2}$ (c) $\displaystyle\sum_{n=1}^{\infty} \sin^2\left(\frac{1}{n}\right)$