Section 12.4: The Cross Product

Problem 1. Let $\mathbf{a} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$ and $\mathbf{b} = -\mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$. Determine each of the following:
(a) $\mathbf{a} \times \mathbf{b}$ (b) $|\mathbf{a} \times \mathbf{b}|$ (c) two unit vectors that are orthogonal to $\mathbf{a}$ and $\mathbf{b}$

(a) We have

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -3 \\ -1 & 2 & 5 \end{vmatrix} = \begin{vmatrix} 2 & -3 \\ 2 & 5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & -3 \\ -1 & 5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \mathbf{k}$$
$$= [2(5) - (-3)(2)]\,\mathbf{i} - [1(5) - (-3)(-1)]\,\mathbf{j} + [1(2) - 2(-1)]\,\mathbf{k}$$
$$= 16\mathbf{i} - 2\mathbf{j} + 4\mathbf{k} = \langle 16, -2, 4 \rangle.$$

(b) The magnitude of $\mathbf{a} \times \mathbf{b}$ is

$$|\mathbf{a} \times \mathbf{b}| = \sqrt{16^2 + (-2)^2 + 4^2} = \sqrt{276} = 2\sqrt{69}.$$

(c) The vectors $\mathbf{a} \times \mathbf{b}$ and $-\mathbf{a} \times \mathbf{b}$ are both orthogonal to $\mathbf{a}$ and $\mathbf{b}$. Note, though, they are not unit vectors, as their magnitude is not 1 (by part (b)). However, the vectors

$$\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \frac{1}{2\sqrt{69}} \langle 16, -2, 4 \rangle = \left\langle \frac{8}{\sqrt{69}}, \frac{-1}{\sqrt{69}}, \frac{2}{\sqrt{69}} \right\rangle$$

and

$$-\frac{\mathbf{a} \times \mathbf{b}}{|\mathbf{a} \times \mathbf{b}|} = \left\langle \frac{-8}{\sqrt{69}}, \frac{1}{\sqrt{69}}, \frac{-2}{\sqrt{69}} \right\rangle$$

are unit vectors that are orthogonal to both $\mathbf{a}$ and $\mathbf{b}$.

Problem 2.
(a) Find a nonzero vector orthogonal to the plane through the points $P = (-2, 0, 4)$, $Q = (1, 3, -2)$, and $R = (0, 3, 5)$.
(b) Find the area of the triangle PQR.

(a) Since the plane through the points $P = (-2, 0, 4)$, $Q = (1, 3, -2)$, and $R = (0, 3, 5)$ contains the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$, the cross product of $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ is a vector that is orthogonal to these vectors, and hence, to the plane through the points P, Q, and R (the vectors $\overrightarrow{PQ}$ and $\overrightarrow{PR}$ lie on this plane).

Since

$$\overrightarrow{PQ} = \langle 1 - (-2)\,, 3 - 0\,, -2 - 4 \rangle = \langle 3\,, 3\,, -6 \rangle$$

and

$$\overrightarrow{PR} = \langle 0 - (-2)\,, 3 - 0\,, 5 - 4 \rangle = \langle 2\,, 3\,, 1 \rangle$$

Then

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 3 & -6 \\ 2 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 3 & -6 \\ 3 & 1 \end{vmatrix}\mathbf{i} - \begin{vmatrix} 3 & -6 \\ 2 & 1 \end{vmatrix}\mathbf{j} + \begin{vmatrix} 3 & 3 \\ 2 & 3 \end{vmatrix}\mathbf{k}$$
$$= [3(1)-(-6)(3)]\,\mathbf{i} - [3(1)-(-6)(2)]\,\mathbf{j} + [3(3)-3(2)]\,\mathbf{k}$$
$$= 21\mathbf{i} - 15\mathbf{j} + 3\mathbf{k} = \langle 21, -15, 3\rangle.$$

is a nonzero vector that is orthogonal to the plane through the points P, Q, and R.

(b) The area of the triangle determined by P, Q, and R is equal to half the area of the parallelogram determined by the three points. Using part (a), the area of the parallelogram is

$$|\overrightarrow{PQ}\times\overrightarrow{PR}| = |\langle 21, -15, 3\rangle| = \sqrt{21^2+(-15)^2+3^2} = \sqrt{675},$$

and so the area of the triangle $\triangle PQR$ is $\frac{1}{2}\sqrt{675}$.

Problem 3. Find the volume of the parallelepiped with adjacent edges PQ, PR, and PS if $P=(-2,1,0)$, $Q=(2,3,2)$, and $R=(1,4,-1)$, and $S=(3,6,1)$.

We can view the adjacent edges PQ, PR, and PS of the parallelepiped as displacement vectors:

$$\overrightarrow{PQ} = \langle 2-(-2)\,, 3-1\,, 2-0\rangle = \langle 4\,,2\,,2\rangle$$

and

$$\overrightarrow{PR} = \langle 1-(-2)\,, 4-1\,, -1-0\rangle = \langle 3\,,3\,,-1\rangle.$$

Since

$$\overrightarrow{PS} = \langle 3-(-2)\,, 6-1\,, 1-0\rangle = \langle 5\,,5\,,1\rangle.$$

The volume of the parallelepiped with adjacent edges PQ, PR, and PS is the scalar triple product

$$\overrightarrow{PQ}\cdot(\overrightarrow{PR}\times\overrightarrow{PS}) = \langle 4\,,2\,,2\rangle\cdot(\langle 3\,,3\,,-1\rangle\times\langle 5\,,5\,,1\rangle)$$
$$= \langle 4\,,2\,,2\rangle\cdot\langle 8\,,-8\,,0\rangle$$
$$= 4(8)+2(-8)+2(0)$$
$$= 16.$$

Section 12.5: Equations of Lines and Planes

Problem 4. Find the parametric equations of the line L through the point $(6,0,-2)$ and parallel to the line

$$x = 4-3t, \quad y=-1+4t, \quad z = 6+5t.$$

HINT: Determine the direction vector $\mathbf{v}$ of the line whose parametric equations are given.

Since the line L is parallel to the line whose parametric equations are given, we can use the direction vector $\mathbf{v} = \langle -3, 4, 5\rangle$ of the given line as the direction vector of the line L. Since L

passes through the point $(6,0,-2)$, its parametric equations are

$$x = 6 - 3t, \quad y = 4t, \quad z = -2 + 5t.$$

Problem 5. Find the symmetric equations of the line L through the point $(2,1,0)$ that is perpendicular to the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

HINT: The direction vector the the line should be perpendicular to both $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$.

As noted in the hint, the direction vector $\mathbf{v}$ of L is perpendicular to each of the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{j}+\mathbf{k}$. Then, for example, we can choose $\mathbf{v} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k})$. Then since $\mathbf{i}+\mathbf{j} = \langle 1,1,0\rangle$ and $\mathbf{j}+\mathbf{k} = \langle 0,1,1\rangle$, we have

$$\begin{aligned}\mathbf{v} = (\mathbf{i}+\mathbf{j}) \times (\mathbf{j}+\mathbf{k}) = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{vmatrix} &= \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} \mathbf{k} \\ &= [1(1) - 0(1)]\,\mathbf{i} - [1(1) - 0(0)]\,\mathbf{j} + [1(1) - 1(0)]\,\mathbf{k} \\ &= \mathbf{i} - \mathbf{j} + \mathbf{k} \\ &= \langle 1,-1,1\rangle.\end{aligned}$$

Therefore, our line's direction vector is $\mathbf{v} = \langle 1,-1,1\rangle$.

Then the parametric equations of the line L are

$$x = 2 + t, \quad y = 1 - t, \quad z = t.$$

By solving for t is each of our parametric equations, we obtain our symmetric equations, which are

$$t = x - 2 = -y + 1 = z.$$

Problem 6. (a) Find the point of intersection of the line in Problem 4 and the yz-plane.
(b) Find the point of intersection of the line in Problem 5 and the plane $x + y = 1$.

(a) The parametric equations we found of the line in Problem 1 are

$$x = 6 - 3t, \quad y = 4t, \quad z = -2 + 5t.$$

Since the point of intersection of the line and the yz-plane must have the x-coordinate equal to zero, then

$$x = 6 - 3t \quad \Rightarrow \quad t = 2.$$

This means that the point of intersection corresponds to when the parameter $t = 2$. Then the y-entry of the point is $y = 4(2) = 8$ and the z-entry is $z = -2 + 5(2) = 8$, so the point of intersection is $(0,8,8)$.

(b) The parametric equations we found of the line in Problem 2 are

$$x = 2 + t, \quad y = 1 - t, \quad z = t.$$

Since the point of intersection of the line and the plane $x + y = 1$ must have its y-coordinate satisfy $y = -x + 1$ and $y = 1 - t$, then

$$-x + 1 = 1 - t \quad \Rightarrow \quad x = t.$$

Additionally, since the x-coordinate of the point must satisfy $x = 2 + t$, then

$$t = 2 + t \quad \Rightarrow \quad 0 = 2,$$

which does not make sense! This means that the point of intersection between the line from Problem 2 and the plane $x + y = 1$ does not exist.

Problem 7. Show that the lines L_1 and L_2 with parametric equations

$$L_1: \quad x = 1 + t, \quad y = -2 + 3t, \quad z = 4 - t,$$

$$L_2: \quad x = 2s, \quad y = 3 + s, \quad z = -3 + 4s,$$

are **skew lines**; that is, they are not parallel and do not intersect (and therefore do not lie on the same plane).

HINT: If the lines are parallel, then their direction vectors are parallel. If the lines intersect, then they have a point (x, y, z) in common (the given parametric equations give you formulas for the entries of any point on the lines).

If the two lines L_1 and L_2 are parallel, then their direction vectors are parallel. A direction vector of L_1 is $\mathbf{v}_1 = \langle 1, 3, -1 \rangle$ and a direction vector of L_2 is $\mathbf{v}_2 = \langle 2, 1, 4 \rangle$. Since the angle θ between $\mathbf{v}_1$ and $\mathbf{v}_2$ is

$$\theta = \cos^{-1}\left(\frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{|\mathbf{v}_1|\,|\mathbf{v}_2|}\right) = \cos^{-1}\left(\frac{1 \cdot 2 + 3 \cdot 1 + (-1)4}{\sqrt{1^2 + 3^2 + (-1)^2}\sqrt{2^2 + 1^2 + 4^2}}\right) = \cos^{-1}\left(\frac{1}{\sqrt{11}\sqrt{21}}\right) \approx 86^\circ,$$

then $\mathbf{v}_1$ and $\mathbf{v}_2$ are **not parallel**.

Let us show that L_1 and L_2 do not intersect. We will do this by assuming that they do intersect and showing that this is impossible. Suppose L_1 and L_2 intersect at some point $P = (x, y, z)$. Then by the parametric equations of our lines, we must have

$$1 + t = 2s, \qquad -2 + 3t = 3 + s, \qquad \text{and} \qquad 4 - t = -3 + 4s.$$

If we solve for t and s using the first two equations, we obtain $t = 11/5$ and $s = 8/5$. However, by substituting these values into the z equations we have $4 - 11/5 = 9/5$ is equal to $-3 + 4(8/5) = 17/5$, which is false. Therefore, L_1 **and** L_2 **cannot share a point of intersection.**

Since L_1 and L_2 **are not parallel and they do not intersect, then they must be skew**.

Problem 8. Find a vector equation for the line segment from the point $(6, -1, 9)$ to the point $(7, 6, 0)$.

HINT: Remember that the vector equation through the vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ is
$\mathbf{r} = (1-t)\mathbf{r}_0 + t\mathbf{r}_1$, where $0 \leq t \leq 1$.

We begin by obtaining the two position vectors $\mathbf{r}_0$ and $\mathbf{r}_1$ of the given points. We have $\mathbf{r}_0 = \langle 6, -1, 9\rangle$ and $\mathbf{r}_1 = \langle 7, 6, 0\rangle$. Then the line segment from $\mathbf{r}_0$ to $\mathbf{r}_1$ is

$$\begin{aligned}\mathbf{r} &= (1-t)\mathbf{r}_0 + t\mathbf{r}_1\\ &= (1-t)\langle 6, -1, 9\rangle + t\langle 7, 6, 0\rangle\\ &= \langle 6, -1, 9\rangle - t\langle 6, -1, 9\rangle + t\langle 7, 6, 0\rangle\\ &= \langle 6, -1, 9\rangle + t\langle -6+7, 1+6, -9+0\rangle\\ &= \langle 6, -1, 9\rangle + t\langle 1, 7, -9\rangle,\end{aligned}$$

where $0 \leq t \leq 1$. That is, the vector equation is $\mathbf{r} = \langle 6, -1, 9\rangle + t\langle 1, 7, -9\rangle$, where $0 \leq t \leq 1$.

Problem 9. Find an equation of the plane that contains the line $x = 1 + t$, $y = 2 - t$, $z = 4 - 3t$ and is parallel to the plane $5x + 2y + z = 1$

Two planes are parallel if their normal vectors are parallel to one another. Note that the given plane has a normal vector $\mathbf{n} = \langle 5, 2, 1\rangle$. Since the given line lies on the plane, then its direction vector $\mathbf{v} = \langle 1, -1, -3\rangle$ must be perpendicular to the normal vector $\mathbf{n}$ of the given plane. Let us verify this:

$$\mathbf{n}\cdot\mathbf{v} = 5\cdot 1 + 2(-1) + 1(-3) = 5 - 2 - 3 = 0.$$

Since the line lies on the plane we are looking for, all of its points must lie on the plane. If we choose $t = 0$, we see that $(1, 2, 4)$ is a point on our line and hence, a point on the plane we are seeking. Since parallel vectors are scalar multiples of one another, we can choose $\mathbf{n} = \langle 5, 2, 1\rangle$ to be the normal vector of the plane we are looking for. Then the scalar equation of the new plane is

$$5(x-1) + 2(y-2) + 1(z-4) = 0,$$

or equivalently,

$$5x + 2y + z = 13,$$

which is the linear equation of the plane.

Problem 10. Find an equation of the plane that passes through the point $(3, 5, -1)$ and contains the line $x = 4 - t$, $y = -1 + 2t$, $z = -3t$.

We are given a point $P_0 = (3, 5, -1)$ on the plane, but to obtain an equation for the plane, we need a normal vector. Since the line $x = 4 - t$, $y = -1 + 2t$, $z = -3t$ lies on the plane, the plane contains any point on the line. Let us find two points on the line.

We can find two points on the line (and hence, on the plane) by plugging in $t = 1$ and $t = 2$. When $t = 1$, we obtain the point

$$P_1 = (4-1, -1+2(1), -3(1)) = (3, 1, -3),$$

and when $t = 2$, we obtain the point

$$P_2 = (4 - 2, -1 + 2(2), -3(2)) = (2, 3, -6).$$

Now that we have three points, P_0, P_1, and P_2 on the plane, we can find two vectors, say $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$, both of which lie on the plane. We have

$$\overrightarrow{P_0P_1} = \langle 3 - 3, 1 - 5, -3 - (-1) \rangle = \langle 0, -4, -2 \rangle$$

and

$$\overrightarrow{P_0P_2} = \langle 2 - 3, 3 - 5, -6 - (-1) \rangle = \langle -1, -2, -5 \rangle.$$

A normal vector $\mathbf{n}$ to the plane must be orthogonal to each of the vectors $\overrightarrow{P_0P_1}$ and $\overrightarrow{P_0P_2}$. We can choose $\mathbf{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2}$. We have

$$\begin{aligned}
\mathbf{n} = \overrightarrow{P_0P_1} \times \overrightarrow{P_0P_2} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & -4 & -2 \\ -1 & -2 & -5 \end{vmatrix} &= \begin{vmatrix} -4 & -2 \\ -2 & -5 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 0 & -2 \\ -1 & -5 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 0 & -4 \\ -1 & -2 \end{vmatrix} \mathbf{k} \\
&= 16\mathbf{i} + 2\mathbf{j} - 4\mathbf{k} \\
&= \langle 16, 2, -4 \rangle.
\end{aligned}$$

Then the equation of the plane is

$$16(x - 3) + 2(y - 5) - 4(z + 1) = 0,$$

or

$$16x + 2y - 4z = 62,$$

or

$$8x + y - 2z = 31.$$