MAT 1505 (Dr. Fuentes) Worksheet 5 - Section 6.5 & 7.1

Section 6.5: Average Value of a Function

Problem 1.
(a) Find the average value f_{avg} of the function $f(x) = (x+1)^3$ on the interval $[0,2]$.

(b) Use the Mean Value for Integrals to find a value c in $[0,2]$ such that $f(c) = f_{\text{avg}}$.

(a) Let $u = x + 1$. Then $du = dx$, and when $x = 0$, $u = 1$ and when $x = 2$, $u = 3$. Then the average value of f over $[0,2]$ is

$$f_{\text{avg}} = \frac{1}{2-0}\int_0^2 (x+1)^3\,\mathrm{dx} = \frac{1}{2}\int_1^3 u^3\,\mathrm{du} = \frac{1}{2}\left(\frac{1}{4}u^4\right)\Big]_1^3 = \frac{1}{8}\left(3^4 - 1^4\right) = \frac{1}{8}\cdot 80 = 10.$$

(b) Since $f(x) = (x+1)^3$ is a polynomial, it is continuous everywhere and hence, on $[0,2]$. Then by the MVT for integrals, there exists a number c such that $0 < c < 2$ and $f(c) = 10$. We solve for c in the following equation:

$$(c+1)^3 = 10 \iff c + 1 = \sqrt[3]{10} \iff c = \sqrt[3]{10} - 1 \approx 1.15.$$

Therefore, the value c in $[0,2]$ that satisfies the conclusion of the MVT for integrals is $c = \sqrt[3]{10} - 1$.

Section 7.1: Integration by Parts

Recall the formula for integration by parts:

$$\int \mathbf{f}(\mathbf{x})\mathbf{g}'(\mathbf{x})\,\mathbf{dx} = \mathbf{f}(\mathbf{x})\mathbf{g}(\mathbf{x}) - \int \mathbf{g}(\mathbf{x})\mathbf{f}'(\mathbf{x})\,\mathbf{dx}.$$

If we let $u = f(x)$ and $v = g(x)$ then the differentials

$$\mathbf{du} = \mathbf{f}'(\mathbf{x})\,\mathrm{dx} \qquad \text{and} \qquad \mathrm{dv} = \mathbf{g}'(\mathbf{x})\,\mathrm{dv}.$$

Then the integration by parts formula can be rewritten as

$$\int \mathbf{u}\,\mathbf{dv} = \mathbf{uv} - \int \mathbf{v}\,\mathbf{du}.$$

Problem 2. Evaluate $\int \ln(x)\,\mathrm{dx}$.

We choose

$$u = \ln(x) \qquad \text{and} \qquad \mathrm{dv} = \mathrm{dx}.$$

Then

$$\mathrm{du} = \frac{1}{x}\,\mathrm{dx} \qquad \text{and} \qquad \mathrm{v} = \int 1\,\mathrm{dv} = \int 1\mathrm{dx} = x.$$

Then by the IBP formula we have

$$\int \ln(x)\,\mathrm{dx} = \int u\,\mathrm{dv} = uv - \int v\,\mathrm{du} = \ln(x)\,x - \int x\frac{1}{x}\,\mathrm{dx} = \ln(x)\,x - \int 1\,\mathrm{dx} = \ln(x)\,x - x + C.$$

Problem 3. Evaluate $\int t^2 e^t \, \mathrm{dt}$. **Hint:** You will need a second application of integration by parts.

Since t^2 has a "simpler" derivative than e^t, we choose

$$u = t^2 \qquad \text{and} \qquad \mathrm{dv} = e^t \, \mathrm{dt}.$$

Then

$$\mathrm{du} = 2t^2 \, \mathrm{dt} \qquad \text{and} \qquad \mathrm{v} = \int 1 \, \mathrm{dv} = \int e^t \, \mathrm{dt} = e^t.$$

Then by the IBP formula we have

$$\int t^2 e^t \, \mathrm{dt} = \int u \, \mathrm{dv} = uv - \int v \, \mathrm{du} = t^2 e^t - \int e^t 2t \, \mathrm{dt} = t^2 e^t - 2 \int t e^t \, \mathrm{dt} \tag{1}$$

We will need another application of IBP for the integral $\int t e^t \, \mathrm{dt}$.

Since t has a "simpler" derivative than e^t, we choose

$$u^* = t \qquad \text{and} \qquad \mathrm{d}v^* = e^t \, \mathrm{dt}.$$

$$\mathrm{d}u^* = \mathrm{dt} \qquad \text{and} \qquad v^* = \int 1 \, \mathrm{d}v^* = \int e^t \, \mathrm{dt} = e^t.$$

Then by the IBP formula we have

$$\int t e^t \, \mathrm{dt} = \int u^* \, \mathrm{d}v^* = u^* v^* - \int v^* \, \mathrm{d}u^* = t e^t - \int e^t \, \mathrm{dt} = t e^t - e^t + C^*, \tag{2}$$

where C^* is a constant. Then by substituting the rightmost side of (2) into the rightmost side of (1), we obtain

$$\int t^2 e^t \, \mathrm{dt} = t^2 e^t - 2 \int t e^t \, \mathrm{dt} = t^2 e^t - 2 \left(t e^t - e^t + C^* \right) = t^2 e^t - 2t e^t - 2e^t + C,$$

where $C = 2C^*$, a constant.

Problem 4. Evaluate $\int e^x \sin(x) \, \mathrm{dx}$.

Note: In this case, we may choose $u = e^x$ and $\mathrm{dv} = \sin(x) \, \mathrm{dx}$ or $u = \sin(x)$ and $\mathrm{dv} = e^x \, \mathrm{dx}$. We will show the solution for the first option.

Let us choose

$$u = e^x \qquad \text{and} \qquad \mathrm{dv} = \sin(x) \, \mathrm{dx}.$$

Then

$$\mathrm{du} = e^x \, \mathrm{dx} \qquad \text{and} \qquad \mathrm{v} = \int 1 \, \mathrm{dv} = \int \sin(x) \, \mathrm{dx} = -\cos(x).$$

Then by the IBP formula we have

$$\int e^x \sin(x) \, \mathrm{dx} = \int u \, \mathrm{dv} = uv - \int v \, \mathrm{du} = -e^x \cos(x) + \int e^x \cos(x) \, \mathrm{dx} \tag{3}$$

We will need another application of IBP for the integral $\int e^x \cos(x) \, \mathrm{dx}$.

we choose

$$u^* = e^x \qquad \text{and} \qquad \mathrm{d}v^* = \cos(x) \, \mathrm{dx}.$$

$$du^* = e^x\,\mathrm{dx} \qquad \text{and} \qquad v^* = \int 1\,\mathrm{d}v^* = \int \cos(x)\,\mathrm{dx} = \sin(x).$$

Then by the IBP formula we have

$$\int e^x \cos(x)\,\mathrm{dx} = \int u^*\,\mathrm{d}v^* = u^*v^* - \int v^*\,\mathrm{d}u^* = e^x \sin(x) - \int e^x \sin(x)\,\mathrm{dx}. \tag{4}$$

By substituting the rightmost side of (4) into the rightmost side of (3), we obtain

$$\int e^x \sin(x)\,\mathrm{dx} = -e^x \cos(x) + \int e^x \cos(x)\,\mathrm{dx} = -e^x \cos(x) + e^x \sin(x) - \int e^x \sin(x)\,\mathrm{dx}.$$

Adding $\int e^x \sin(x)\,\mathrm{dx}$ to both sides gives us

$$2\int e^x \sin(x)\,\mathrm{dx} = -e^x \cos(x) + e^x \sin(x).$$

Therefore,

$$\int e^x \sin(x)\,\mathrm{dx} = -\frac{1}{2}e^x \cos(x) + \frac{1}{2}e^x \sin(x) + C,$$

for some constant C.

Integration by Parts for Definite Integrals

Note that

$$\int_a^b [f(x)g'(x) + g(x)f'(x)]\,\mathrm{dx} = f(x)g(x)\Big]_a^b,$$

which is equivalent to

$$\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{f(x)g'(x)\,dx} = \mathbf{f(x)g(x)}\Big]_{\mathbf{a}}^{\mathbf{b}} - \int_{\mathbf{a}}^{\mathbf{b}} \mathbf{g(x)f'(x)\,dx}.$$

Problem 5. Evaluate $\int_0^1 \tan^{-1}(x)\,\mathrm{dx}$. **Hint:** After one application of integration by parts, you will need to use substitution for the new integral you obtain.

Let

$$u = \tan^{-1}(x) \qquad \text{and} \qquad \mathrm{dv} = \mathrm{dx}.$$

Then

$$\mathrm{du} = \frac{1}{x^2+1}\,\mathrm{dx} \qquad \text{and} \qquad \mathrm{v} = \int 1\,\mathrm{dv} = \int 1\mathrm{dx} = x.$$

Then by the IBP formula we have

$$\begin{aligned}\int_0^1 \tan^{-1}(x)\,\mathrm{dx} = \int_0^1 u\,\mathrm{dv} = uv\Big]_0^1 - \int_0^1 v\,\mathrm{du} &= x\,\tan^{-1}(x)\Big]_0^1 - \int_0^1 \frac{x}{x^2+1}\,\mathrm{dx} \\ &= 1\cdot\tan^{-1}(1) - 0\cdot\tan^{-1}(0) - \int_0^1 \frac{x}{x^2+1}\,\mathrm{dx} \\ &= \pi/4 - \int_0^1 \frac{x}{x^2+1}\,\mathrm{dx}. \end{aligned} \tag{5}$$

To evaluate the integral $\int_0^1 \frac{x}{x^2+1}\,\mathrm{dx}$ we use the substitution $s = x^2 + 1$. Then $\mathrm{ds} = 2x\,\mathrm{dx}$, or equivalently, $(1/2)\mathrm{ds} = x\,\mathrm{dx}$. When $x = 0$, $s = 0^2 + 1 = 1$ and when $x = 1$, $s = 1^2 + 1 = 2$. Then

$$\int_0^1 \frac{x}{x^2+1}\,\mathrm{dx} = \frac{1}{2}\int_1^2 \frac{1}{s}\,\mathrm{ds} = \frac{1}{2}\ln(|s|)\Big]_1^2 = \frac{1}{2}\left(\ln(2) - \ln(1)\right) = \frac{1}{2}\ln(2). \tag{6}$$

By substituting the rightmost side of (6) into the last part of (5), we obtain

$$\int_0^1 \tan^{-1}(x)\,\mathrm{dx} = \pi/4 - \frac{1}{2}\ln(2).$$