Section 12.2: Vectors

Problem 1. Find the vector in $\mathbb{R}^3$ that has the opposite direction as $\langle 6, 2, -3\rangle$ and has length 4.

Let $\mathbf{a} = \langle 6, 2, -3\rangle$. Recall that the unit vector $\mathbf{u} = \dfrac{\mathbf{a}}{|\mathbf{a}|}$ has the same direction as the vector $\mathbf{a}$ and has length 1 . Note that for any scalar c,

$$|c\mathbf{u}| = |c|\,|\mathbf{u}| = |c|.$$

Thus, if $c = -4$, the direction of $4\mathbf{u}$ is opposite the direction of $\mathbf{u}$, and hence $\mathbf{a}$, and its magnitude is 4. We can explicitly find what $4\mathbf{u}$ is! We have

$$-4\mathbf{u} = \frac{-4}{|\mathbf{a}|}\mathbf{a} = \frac{-4}{\sqrt{6^2+2^2+(-3)^2}}\langle 6, 2, -3\rangle = \frac{-4}{7}\langle 6, 2, -3\rangle = \langle -\frac{24}{7}, -\frac{8}{7}, \frac{12}{7}\rangle.$$

Section 12.3: The Dot Product

Problem 2. Use vectors to determine whether the triangle with vertices $P = (1, -3, -2)$, $Q = (2, 0, -4)$, and $R = (6, -2, -5)$ is a right triangle.

Note that the right angle of a right triangle lies across the hypotenuse, the longest side. Since

$$|PQ| = \sqrt{(2-1)^2 + (0-(-3))^2 + (-4-(-2))^2} = \sqrt{1+9+4} = \sqrt{14},$$

$$|PR| = \sqrt{(6-1)^2 + (-2-(-3))^2 + (-5-(-2))^2} = \sqrt{25+1+9} = \sqrt{35},$$

and

$$|QR| = \sqrt{(6-2)^2 + (-2-0)^2 + (-5-(-4))^2} = \sqrt{16+4+1} = \sqrt{21},$$

the longest side of the triangle $\triangle PQR$ is PR. Then if $\triangle PQR$ is indeed a right triangle, its right angle would be the angle between the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$.

We must determine the coordinates of the displacement vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$. We have

$$\overrightarrow{QP} = \langle 1-2\,, -3-0\,, -2-(-4)\rangle = \langle -1\,, -3\,, 2\rangle$$

and

$$\overrightarrow{QR} = \langle 6-2\,, -2-0\,, -5-(-4)\rangle = \langle 4\,, -2\,, -1\rangle.$$

Since

$$\overrightarrow{QP}\cdot\overrightarrow{QR} = \langle -1\,, -3\,, 2\rangle \cdot \langle 4\,, -2\,, -1\rangle = -4+6-2 = 0,$$

the vectors $\overrightarrow{QP}$ and $\overrightarrow{QR}$ are perpendicular, that is, the angle between the sides QP and QR is a right angle. Therefore, $\triangle PQR$ is a right triangle.

Problem 3. In class, we learned that vectors $\mathbf{a}$ and $\mathbf{b}$ are orthogonal (perpendicular) if and only if $\mathbf{a} \bullet \mathbf{b} = 0$.

Vectors $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if the angle θ between them is 0 or π. That is, $\mathbf{a}$ and $\mathbf{b}$ are parallel if and only if

$$\mathbf{a} \bullet \mathbf{b} = |\mathbf{a}| \cdot |\mathbf{b}| \cdot \cos(\theta) = |\mathbf{a}| \cdot |\mathbf{b}|(\pm 1) \quad \Leftrightarrow \quad \frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \pm 1,$$

since $\cos(0) = 1$ and $\cos(\pi) = -1$.

Let $\mathbf{a} = \langle 1, 2, 3 \rangle$, $\mathbf{b} = \langle 0, 1, 3 \rangle$, and $\mathbf{c} = \langle 2, -1, -1 \rangle$. Determine whether the following pairs vectors are orthogonal, parallel, or neither.
(a) $\mathbf{a}$, $\mathbf{b}$ (b) $\mathbf{a}$, $\mathbf{c}$, (c) $\mathbf{a}$, $\mathbf{b} + \mathbf{c}$ (d) $2\mathbf{a}$, $\mathbf{b}$.

(a) We have

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, 3 \rangle \bullet \langle 0, 1, 3 \rangle = 1(0) + 2(1) + 3(3) = 0 + 2 + 9 = 11.$$

Then

$$\frac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} = \frac{11}{\sqrt{14}\sqrt{10}} \neq \pm 1.$$

Since $\mathbf{a} \cdot \mathbf{b} \neq 0$ and $\dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} \neq \pm 1$, the vectors are neither orthogonal (perpendicular) nor parallel.

Another approach: find the angle θ between $\mathbf{a}$ and $\mathbf{b}$. We have

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{a}|\,|\mathbf{b}|}\right) = \cos^{-1}\left(\frac{11}{\sqrt{14}\sqrt{10}}\right) \approx 21.6^\circ.$$

Since the angle between $\mathbf{a}$ and $\mathbf{b}$ is not 90°, 0°, nor 180°, we see that the vectors are neither orthogonal nor parallel.

(b) We have

$$\mathbf{a} \cdot \mathbf{b} = \langle 1, 2, 3 \rangle \bullet \langle 2, -1, -1 \rangle = 1(2) + 2(-1) + 3(-1) = 2 - 2 - 3 = -3.$$

Then

$$\frac{\mathbf{a} \bullet \mathbf{c}}{|\mathbf{a}| \cdot |\mathbf{c}|} = \frac{-3}{\sqrt{14}\sqrt{6}} \neq \pm 1.$$

Since $\mathbf{a} \cdot \mathbf{b} \neq 0$ and $\dfrac{\mathbf{a} \bullet \mathbf{b}}{|\mathbf{a}| \cdot |\mathbf{b}|} \neq \pm 1$, the vectors are neither orthogonal (perpendicular) nor parallel.

Another approach: find the angle θ between $\mathbf{a}$ and $\mathbf{c}$. We have

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot \mathbf{c}}{|\mathbf{a}|\,|\mathbf{c}|}\right) = \cos^{-1}\left(\frac{-3}{\sqrt{14}\sqrt{6}}\right) \approx 109.1^\circ.$$

Since the angle between $\mathbf{a}$ and $\mathbf{b}$ is not $90°$, $0°$, nor $180°$, we see that the vectors are neither orthogonal nor parallel.

(c) Using properties of dot products and our answers from parts (a) and (b), we have

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c} = 11 - 3 = 8.$$

We have $\mathbf{b} + \mathbf{c} = \langle 2, 0, 2 \rangle$, so

$$\frac{\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})}{|\mathbf{a}|\,|\mathbf{b} + \mathbf{c}|} = \frac{8}{\sqrt{14}\sqrt{8}} \neq \pm 1.$$

Another approach: find the angle θ between $\mathbf{a}$ and $\mathbf{b} + \mathbf{c}$. We have

$$\theta = \cos^{-1}\left(\frac{\mathbf{a} \cdot (\mathbf{b} + \mathbf{c})}{|\mathbf{a}|\,|\mathbf{b} + \mathbf{c}|}\right) = \cos^{-1}\left(\frac{8}{\sqrt{14}\sqrt{8}}\right) \approx 40.9°.$$

Since the angle between $\mathbf{a}$ and $\mathbf{b}$ is not $90°$, $0°$, nor $180°$, we see that the vectors are neither orthogonal nor parallel.

(d) Since $2\mathbf{a}$ is a positive scalar multiple of $\mathbf{a}$, the angle between $2\mathbf{a}$ and $\mathbf{b}$ is the same as the angle between $\mathbf{a}$ and $\mathbf{b}$. In part (a), we determined that $\mathbf{a}$ and $\mathbf{b}$ are neither orthogonal nor parallel. Therefore, $2\mathbf{a}$ and $\mathbf{b}$ are neither orthogonal nor parallel.

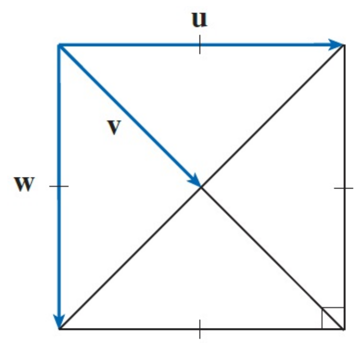
Problem 4. Which of the following expressions are meaningful? Which are meaningless? Explain.
(a) $(\mathbf{a} \bullet \mathbf{b}) \bullet \mathbf{c}$, (b) $|\mathbf{a}|(\mathbf{a} \bullet \mathbf{c})$, (c) $\mathbf{a} \bullet \mathbf{b} + \mathbf{c}$

(a) Note that $\mathbf{a} \bullet \mathbf{b}$ is a scalar, since it is a dot product. The expression is meaningless since we cannot take the dot product of a scalar, $\mathbf{a} \bullet \mathbf{b}$ and a vector $\mathbf{c}$, as the dot product is only defined for two vectors.

(b) The expression has meaning since it is the product of two scalars: $|\mathbf{a}|$, the magnitude of $\mathbf{a}$, and $\mathbf{a} \cdot \mathbf{c}$, a dot product.

(c) The expression is meaningless since it consists of the sum of a scalar, $\mathbf{a} \bullet \mathbf{b}$, and a vector, $\mathbf{c}$. Scalars can only be added to scalars, and vectors can only be added to vectors. Note that $\mathbf{a} \bullet \mathbf{b} + \mathbf{c} \neq \mathbf{a} \bullet (\mathbf{b} + \mathbf{c})$, since by the distributive property for vectors, $\mathbf{a} \bullet (\mathbf{b} + \mathbf{c}) = \mathbf{a} \bullet \mathbf{b} + \mathbf{a} \bullet \mathbf{c}$.

Problem 5. If $\mathbf{u}$ is a unit vector, find $\mathbf{u} \bullet \mathbf{v}$ and $\mathbf{u} \bullet \mathbf{w}$.
NOTE: The dashes on the sides of the figure below indicate the figure is a square. Assume that the angles between the diagonal lines are each $\pi/2$ (90 degrees).



Since $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, then $\mathbf{u} \bullet \mathbf{w} = 0$. The angle between $\mathbf{u}$ and $\mathbf{v}$ is $45°$. Then

$$\mathbf{u} \bullet \mathbf{v} = |\mathbf{u}| \cdot |\mathbf{v}| \cdot \cos(45°) = 1 \cdot |\mathbf{v}| \cdot \frac{\sqrt{2}}{2}.$$

Note that the magnitude of $\mathbf{v}$ is half of the length of the diagonal of the square. Since the sides of the square are all of length 1, then

$$|\mathbf{v}| = \frac{\text{length of the diagonal of the square}}{2} = \frac{\sqrt{1^2+1^2}}{2} = \frac{\sqrt{2}}{2}.$$

Then $\mathbf{u} \bullet \mathbf{v} = |\mathbf{v}|(\sqrt{2}/2) = (\sqrt{2}/2)(\sqrt{2}/2) = 1/2$.

Problem 6. Find a unit vector that is orthogonal to both of the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$.

If $\mathbf{u} = \langle u_1, u_2, u_3 \rangle$ is orthogonal to $\mathbf{i}+\mathbf{j} = \langle 1,1,0 \rangle$ and $\mathbf{i}+\mathbf{k} = \langle 1,0,1 \rangle$, then

$$0 = \mathbf{u} \bullet (\mathbf{i}+\mathbf{j}) = u_1 + u_2 \quad \text{and} \quad 0 = \mathbf{u} \bullet (\mathbf{i}+\mathbf{k}) = u_1 + u_3.$$

By solving for u_1 in each of the equations above, we have $-u_2 = u_1 = -u_3$. Then $u_2 = u_3 = -u_1$. Since $\mathbf{u}$ is a unit vector,

$$1 = |\mathbf{u}| = \sqrt{u_1^2+u_2^2+u_3^2} = \sqrt{u_1^2+(-u_1)^2+(-u_1)^2} = \sqrt{3u_1^2} \quad \Rightarrow \quad u_1 = \frac{1}{\sqrt{3}}.$$

Then the vector $\mathbf{u} = \langle \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}} \rangle$ is orthogonal to both of the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$. The vector $-\mathbf{u} = \langle -\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}} \rangle$ is also orthogonal to both of the vectors $\mathbf{i}+\mathbf{j}$ and $\mathbf{i}+\mathbf{k}$.