Section 15.2: Double & Triple Integrals over General Regions

Problem 1. Calculate the following double integrals.

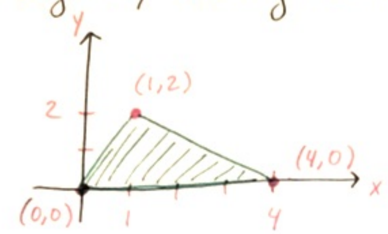
(a) $\displaystyle\iint_D 3xy\,dA$, where D is the triangular region with vertices $(0,0)$, $(1,2)$, and $(4,0)$.

(b) $\displaystyle\int_0^{\pi/2}\int_0^x x\cos(y)\,dy\,dx$

HINT: You will need to apply integration by parts after computing the "inner integral."

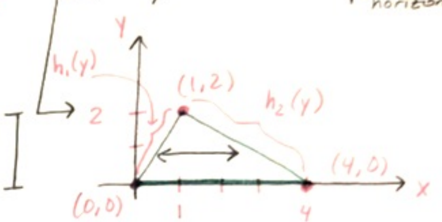
1 (a)

Lets begin by drawing the triangular region D.



We can express this region as TYPE 1 or TYPE 2.

Lets try TYPE 2. (view the region "horizontally")



$2 \geq y \geq 0$

Note that $0 \leq y \leq 2$, and $h_1(y) \leq x \leq h_2(y)$.
We need to determine the eqns. of the lines $x = h_1(y)$ and $x = h_2(y)$.

Using $y = mx+b$ form.

$$y = \frac{2-0}{1-0}x + 0$$
$$\Rightarrow y = 2x$$
$$\Rightarrow x = \tfrac{1}{2}y = h_1(y)$$

Using $y - y_0 = m(x - x_0)$ form.

$$y - 0 = \frac{2-0}{1-4}(x-4)$$
$$y = -\tfrac{2}{3}x + \tfrac{8}{3}$$
$$\Rightarrow 3y = -2x + 8$$
$$\Rightarrow x = -\tfrac{3}{2}y + 4 = h_2(y)$$

Then

$$D = \{(x,y) \mid \tfrac{1}{2}y \leq x \leq -\tfrac{3}{2}y + 4,\ 0 \leq y \leq 2\}$$

Then our integral is

$$\iint_D 3xy\,dA = \int_0^2 \int_{\frac{1}{2}y}^{-\frac{3}{2}y+4} 3xy\,dx\,dy$$

$$= \int_0^2 \left[\frac{3}{2}x^2 y\right]_{x=\frac{1}{2}y}^{x=-\frac{3}{2}y+4} dy$$

$$= \frac{3}{2}\int_0^2 \left(-\tfrac{3}{2}y+4\right)^2 y - \left(\tfrac{1}{2}y\right)^2 y\,dy = \frac{3}{2}\int_0^2 (2y^3 - 12y^2 + 16y)\,dy$$

$$= 3\int_0^2 (y^3 - 6y^2 + 8y)\,dy = 3\left(\frac{y^4}{4} - 2y^3 + 4y^2\right)\Big]_0^2 = \boxed{12}$$

1 (b)

$$\int_0^{\pi/2} \boxed{\int_0^{x} x\cos(y)\,dy}\,dx$$

1st

$$= \int_0^{\pi/2} x\sin(y)\Big]_{y=0}^{y=x} dx = \int_0^{\pi/2} \big(x\sin(x) - x\sin(0)\big)\,dx$$

$$= \int_0^{\pi/2} x\sin(x)\,dx$$

Apply I.B.P.

$u = x \qquad dv = \sin(x)\,dx$

$du = dx \qquad v = -\cos(x)\,dx$

$$\int_0^{\pi/2} u\,dv = uv\Big]_0^{\pi/2} - \int_0^{\pi/2} v\,du$$

=

$$= -x\cos(x)\Big]_0^{\pi/2} - \int_0^{\pi/2} (-\cos(x))\,dx$$

$$= -x\cos(x)\Big]_0^{\pi/2} + \sin(x)\Big]_0^{\pi/2}$$

$$= -\frac{\pi}{2}\cos\left(\frac{\pi}{2}\right) + 0 + \sin\left(\frac{\pi}{2}\right) - \sin(0)$$

$$= 0 + 0 + 1 - 0$$

$$= \boxed{1}$$

Problem 2. Evaluate the following integral by changing the order of integration.

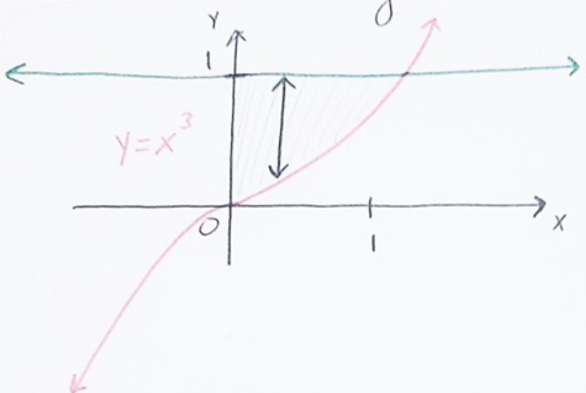
$$\int_0^1 \int_{x^3}^1 y^{2/3} e^y \, dy \, dx$$

The given region of integration is

$$D = \{ (x,y) \mid 0 \le x \le 1, \ x^3 \le y \le 1 \}$$

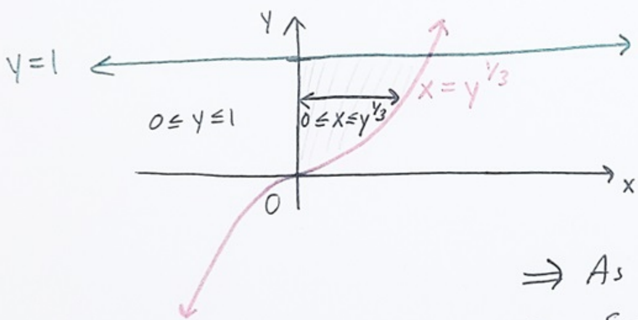
NOTE: D has been expressed as a TYPE 1 region.

Lets draw the region.



* Remember, we view TYPE 1 regions "vertically."

To change the order of integration, we need to convert D to TYPE 2 (view it "horizontally.").



$y = x^3 \Rightarrow x = y^{1/3}$

Notice that

$0 \le y \le 1$ and $0 \le x \le y^{1/3}$

$\Rightarrow$ As a TYPE 2 region,

$$D = \{ (x,y) \mid 0 \le x \le y^{1/3}, \ 0 \le y \le 1 \}$$

Then

$$\int_0^1 \int_{x^3}^1 y^{2/3} e^y \, dy \, dx = \int_0^1 \int_0^{y^{1/3}} y^{2/3} e^y \, dx \, dy$$

$$= \int_0^1 y^{2/3} e^y x \Big]_{x=0}^{x=y^{1/3}} dy$$

Need I.B.P.

$u = y \Rightarrow du = dy$

$dv = e^y dy \Rightarrow v = e^y$

$$= \int_0^1 y e^y dy = y e^y \Big]_0^1 - \int_0^1 e^y dy = e - 0 - e^y \Big]_0^1 = e - (e^1 - e^0)$$

$$= e - e^1 + e^0 = e^0 = \boxed{1}$$

Problem 3. Evaluate the following triple integral.

$$\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z \, dz \, dy \, dx$$

$$\int_0^1 \int_0^1 \int_0^{2-x^2-y^2} xye^z \, dz \, dy \, dx = \int_0^1 \int_0^1 \left[xye^z\right]_{z=0}^{z=2-x^2-y^2} dy \, dx = \int_0^1 \int_0^1 (xye^{2-x^2-y^2} - xy) \, dy \, dx$$

$$= \int_0^1 \left[-\tfrac{1}{2}xe^{2-x^2-y^2} - \tfrac{1}{2}xy^2\right]_{y=0}^{y=1} dx = \int_0^1 \left(-\tfrac{1}{2}xe^{1-x^2} - \tfrac{1}{2}x + \tfrac{1}{2}xe^{2-x^2}\right) dx$$

$$= \left[\tfrac{1}{4}e^{1-x^2} - \tfrac{1}{4}x^2 - \tfrac{1}{4}e^{2-x^2}\right]_0^1 = \tfrac{1}{4} - \tfrac{1}{4} - \tfrac{1}{4}e - \tfrac{1}{4}e + 0 + \tfrac{1}{4}e^2 = \tfrac{1}{4}e^2 - \tfrac{1}{2}e$$

Section 15.3: Double in Polar Coordinates

Problem 4. Evaluate the integral by changing to polar coordinates.

$$\iint_D \sin(x^2 + y^2) \, dA,$$

where D is the region in the first quadrant between the circles with center the origin and radii 1 and 3.

$$\iint_R \sin(x^2+y^2)\,dA = \int_0^{\pi/2} \int_1^3 \sin(r^2)\, r\, dr\, d\theta = \int_0^{\pi/2} d\theta \int_1^3 r\sin(r^2)\, dr = \left[\theta\right]_0^{\pi/2} \left[-\tfrac{1}{2}\cos(r^2)\right]_1^3$$

$$= \left(\tfrac{\pi}{2}\right)\left[-\tfrac{1}{2}(\cos 9 - \cos 1)\right] = \tfrac{\pi}{4}(\cos 1 - \cos 9)$$

Problem 5. Evaluate the integral by changing to polar coordinates.

$$\iint_D \cos(\sqrt{x^2 + y^2}) \, dA,$$

where D is the disk with center the origin and radius 2.

$\iint_D \cos\sqrt{x^2+y^2}\,dA = \int_0^{2\pi} \int_0^2 \cos\sqrt{r^2}\; r\,dr\,d\theta = \int_0^{2\pi} d\theta \int_0^2 r\cos r\,dr$. For the second integral, integrate by parts with $u = r$, $dv = \cos r\,dr$. Then $\iint_D \cos\sqrt{x^2+y^2}\,dA = \left[\theta\right]_0^{2\pi} \left[r\sin r + \cos r\right]_0^2 = 2\pi(2\sin 2 + \cos 2 - 1)$.