Section 14.7: Maximum and Minimum Values

Problem .1 Find the local extreme value(s) and the saddle point(s) of the function, if they exist.
(a) $F(x,y) = xy - x^2y - xy^2$ (b) $f(x,y) = e^x\cos(y)$

(c) $g(x,y) = y^2 - 2y\cos(x), \quad -1 \le x \le 7$

(a) We have
$$F_x = y - 2xy - y^2 \qquad \text{and} \qquad F_y = x - x^2 - 2xy.$$
Solving $F_x = 0$ and $F_y = 0$ gives us
$$0 = y - 2xy - y^2 = y(1-2x-y) \qquad \Rightarrow \qquad y = 0 \text{ or } y = 1-2x.$$
Substituting $y = 0$ into $0 = F_y = x - x^2 - 2xy$ gives us
$$0 = x - x^2 = x(1-x) \qquad \Rightarrow \qquad x = 0 \text{ or } x = 1.$$
Next, substituting $y = 1-2x$ into $0 = F_y = x - x^2 - 2xy$ gives us
$$0 = x - x^2 - 2x(1-2x) = 3x^2 - x = x(3x-1) \qquad \Rightarrow \qquad x = 0 \text{ or } x = 1/3.$$
When $x = 0$, $y = 1-2(0) = 1$ and when $x = 1/3$, $y = 1-2(1/3) = 1/3$. Therefore, the critical points of F are $(0,0)$, $(1,0)$, $(0,1)$, and $(1/3,1/3)$.

Lets compute the second order partial derivatives of F. We have
$$F_{xx} = -2y \qquad F_{xy} = F_{yx} = 1-2x-2y \qquad F_{yy} = -2x.$$
We have
$$D(x,y) = F_{xx}\cdot F_{yy} - F_{xy}^2 = (-2y)(-2x) - (1-2x-2y)^2 = 4xy - (1-2x-2y)^2.$$
Note that
$$D(0,0) = D(1,0) = D(0,1) = -1 < 0,$$
so $(0,0)$, $(1,0)$, and $(0,1)$ are each saddle points. Since
$$D(1/3,\,1/3) = 4(1/3)^2 - (1-2/3-2/3)^2 = 4/9 - 1/9 = 1/3 > 0 \qquad \text{and} \qquad F_{xx}(1/3,\,1/3) = -2/3 < 0,$$
then F has a local maximum at the point $(1/3,1/3)$. The local maximum value at this point is $F(1/3,\,1/3) = (1/3)^2 - (1/3)^3 - (1/3)^3 = 1/27$.

(b) We have
$$f_x = e^x\cos(y) \qquad \text{and} \qquad f_y = -e^x\sin(y).$$
Since $e^x > 0$ for all values of x, then solving $f_x = 0$ and $f_y = 0$ gives us $\cos(y) = 0$ and $\sin(y) = 0$. However, since $\sin(y) = 0$ only when $y = n\pi$ for any integer n, but $\cos(n\pi) \neq 0$ (it can only be ± 1), the system of equations $f_x = 0$ and $f_y = 0$ has no solution, meaning that f has no critical points. Then f has no local extrema nor any saddle points.

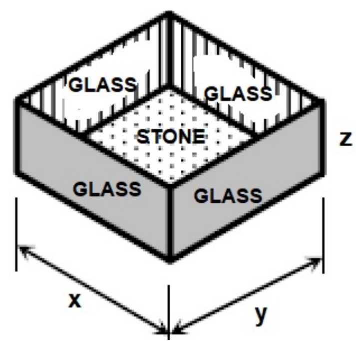
(c)

$f(x,y) = y^2 - 2y\cos x \quad \Rightarrow \quad f_x = 2y\sin x,\ f_y = 2y - 2\cos x,\ f_{xx} = 2y\cos x,\ f_{xy} = 2\sin x,\ f_{yy} = 2.$

Then $f_x = 0$ implies $y = 0$ or $\sin x = 0 \quad \Rightarrow \quad x = 0, \pi,$ or 2π for $-1 \leq x \leq 7$. Substituting $y = 0$ into $f_y = 0$ gives $\cos x = 0 \quad \Rightarrow \quad x = \frac{\pi}{2}$ or $\frac{3\pi}{2}$, substituting $x = 0$ or $x = 2\pi$ into $f_y = 0$ gives $y = 1$, and substituting $x = \pi$ into $f_y = 0$ gives $y = -1$. Thus the critical points are $(0,1)$, $\left(\frac{\pi}{2}, 0\right)$, $(\pi, -1)$, $\left(\frac{3\pi}{2}, 0\right)$, and $(2\pi, 1)$.
$D\left(\frac{\pi}{2}, 0\right) = D\left(\frac{3\pi}{2}, 0\right) = -4 < 0$ so $\left(\frac{\pi}{2}, 0\right)$ and $\left(\frac{3\pi}{2}, 0\right)$ are saddle points.
$D(0,1) = D(\pi,-1) = D(2\pi,1) = 4 > 0$ and
$f_{xx}(0,1) = f_{xx}(\pi,-1) = f_{xx}(2\pi,1) = 2 > 0$, so $f(0,1) = f(\pi,-1) = f(2\pi,1) = -1$ are local minimums.

Problem 2. The base of an aquarium (open top) with volume 800 in^3 is made of stone and the sides are made of glass. If the stone costs five times as much (per in^2) as the glass, find the dimensions of the aquarium that minimize the cost of the materials.

As shown in our picture to the right, the dimensions of the aquarium are x, y, and z. Let C be the cost of the materials to create the aquarium. Let c_s be the cost of the stone per in^2 and let c_g be the cost of the glass per in^2. Then $c_s = 5c_g$ and the cost to create the aquarium is



$$C = c_s xy + c_g z(x+y) = 5c_g xy + c_g 2z(x+y) = c_g(5xy + 2z(x+y)).$$

Let us convert C into a 2-variable function by using the fact that the volume $xyz = 800$. By substituting $z = 800/(xy)$ into C we have

$$C = c_g\left(5xy + 2\frac{800}{xy}(x+y)\right) = c_g(5xy + 1600(y^{-1} + x^{-1})).$$

Our goal is to minimize $C(x,y) = c_g(5xy + 1600(y^{-1} + x^{-1}))$. We have

$$C_x = c_g\left(5y - \frac{1600}{x^2}\right) \qquad \text{and} \qquad C_y = c_g\left(5x - \frac{1600}{y^2}\right).$$

By solving for y in the equation $C_x = 0$, we obtain $y = \dfrac{1600}{5x^2}$. Substituting this into the equation $C_y = 0$ gives us

$$c_g\left(5x - \frac{1600}{\left(\frac{1600}{5x^2}\right)^2}\right) = 0 \Rightarrow 5x - \frac{25x^4}{1600} = 0 \Rightarrow 5x = \frac{25x^4}{1600} \Rightarrow 1600 = 5x^3 \Rightarrow x = \sqrt[3]{\frac{1600}{5}}.$$

Then

$$y = \frac{1600}{5\,(1600/5)^{2/3}} = \sqrt[3]{\frac{1600}{5}}.$$

Let us verify that the function C obtains a minimum value at $(x, y) = \left(\sqrt[3]{\frac{1600}{5}}, \sqrt[3]{\frac{1600}{5}}\right)$. We have

$$D = C_{xx}C_{yy} - C_{xy}^2 = \left(\frac{3200c_g}{x^3}\right)\left(\frac{3200c_g}{y^3}\right) - (5c_g)^2.$$

Then since

$$D\left(\sqrt[3]{\frac{1600}{5}}, \sqrt[3]{\frac{1600}{5}}\right) = \left(\frac{3200c_g}{1600/5}\right)^2 - 25c_g^2 = 100c_g^2 - 5c_g^2 = 95c_g^2 > 0$$

and

$$C_{xx}\left(\sqrt[3]{\frac{1600}{5}}, \sqrt[3]{\frac{1600}{5}}\right) = 100c_g^2 > 0,$$

C does indeed obtain a minimum at $(x, y) = \left(\sqrt[3]{\frac{1600}{5}}, \sqrt[3]{\frac{1600}{5}}\right)$ by the 2nd Derivatives Test. Therefore, the dimensions of the aquarium that minimize its cost are

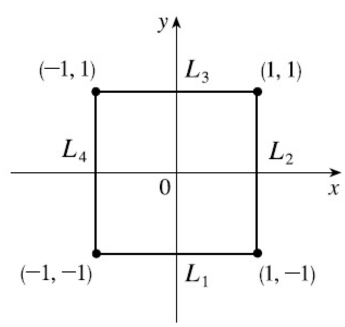
$$x = \sqrt[3]{\frac{1600}{5}} \approx 6.84\,\text{in}, \quad y = \sqrt[3]{\frac{1600}{5}} \approx 6.84\,\text{in}, \quad \text{and} \quad z = \frac{800}{(1600/5)^{2/3}} \approx 17.1\,\text{in}.$$

Problem 3. Find the absolute maximum and minimum values of the function on the given set D.

(a) $f(x, y) = x^2 + y^2 + x^2y + 4, \qquad D = \{(x, y) \mid |x| \le 1, |y| \le 1\}$.

(b) $g(x, y) = xy^2, \qquad D = \{(x, y) \mid x \ge 0, y \ge 0, x^2 + y^2 \le 3\}$.

(a)



$f(x, y) = x^2 + y^2 + x^2y + 4 \;\Rightarrow\; f_x(x, y) = 2x + 2xy$, $f_y(x, y) = 2y + x^2$, and setting $f_x = f_y = 0$ gives $(0, 0)$ as the only critical point in D, with $f(0, 0) = 4$.

On L_1: $y = -1$, $f(x, -1) = 5$, a constant.

On L_2: $x = 1$, $f(1, y) = y^2 + y + 5$, a quadratic in y which attains its maximum at $(1, 1)$, $f(1, 1) = 7$ and its minimum at $\left(1, -\frac{1}{2}\right)$, $f\left(1, -\frac{1}{2}\right) = \frac{19}{4}$.

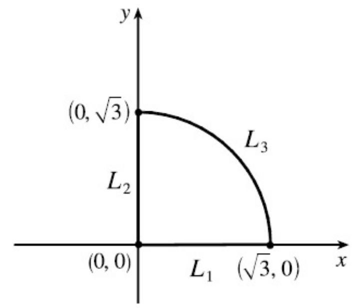
On L_3: $f(x, 1) = 2x^2 + 5$ which attains its maximum at $(-1, 1)$ and $(1, 1)$ with $f(\pm1, 1) = 7$ and its minimum at $(0, 1)$, $f(0, 1) = 5$.

On L_4: $f(-1, y) = y^2 + y + 5$ with maximum at $(-1, 1)$, $f(-1, 1) = 7$ and minimum at $\left(-1, -\frac{1}{2}\right)$, $f\left(-1, -\frac{1}{2}\right) = \frac{19}{4}$.

Thus the absolute maximum is attained at both $(\pm1, 1)$ with $f(\pm1, 1) = 7$ and the absolute minimum on D is attained at $(0, 0)$ with $f(0, 0) = 4$.

(b)

$f(x,y) = xy^2 \quad \Rightarrow \quad f_x = y^2$ and $f_y = 2xy$, and since $f_x = 0 \quad \Leftrightarrow$ $y = 0$, there are no critical points in the interior of D. Along L_1: $y = 0$ and $f(x,0) = 0$. Along L_2: $x = 0$ and $f(0,y) = 0$. Along L_3: $y = \sqrt{3-x^2}$, so let $g(x) = f\left(x, \sqrt{3-x^2}\right) = 3x - x^3$ for $0 \le x \le \sqrt{3}$. Then $g'(x) = 3 - 3x^2 = 0 \quad \Leftrightarrow \quad x = 1$. The maximum value is $f\left(1, \sqrt{2}\right) = 2$ and the minimum occurs both at $x = 0$ and $x = \sqrt{3}$ where $f\left(0, \sqrt{3}\right) = f\left(\sqrt{3}, 0\right) = 0$. Thus the absolute maximum of f on D is $f\left(1, \sqrt{2}\right) = 2$, and the absolute minimum is 0 which occurs at all points along L_1 and L_2.



Section 15.1: Double Integrals over Rectangles

Problem 4. Calculate the following double integrals.

(a) $\displaystyle\iint_R \frac{\ln(y)}{xy}\, dA, \quad$ where $R = [1,3] \times [1,5]$.

(b) $\displaystyle\int_0^1 \int_0^1 \frac{x}{1+xy}\, dy\, dx$

HINT: Use a u-substitution then later you will need to apply integration by parts.

(c) $\displaystyle\iint_R xy\sqrt{x^2+y^2}\, dA, \quad$ where $R = [0,1] \times [0,1]$. **HINT:** Use the u-substitution $u = x^2 + y^2$.

(d) $\displaystyle\int_{-2}^{-1} \int_0^2 \int_0^1 \frac{x^2 e^y}{z}\, dx\, dy\, dz$

(a) In this case, either order of integration is of equal difficulty. We will integrate w.r.t. x first. We have

$$\begin{aligned}
\int_1^5 \int_1^3 \frac{\ln(y)}{xy}\, dx\, dy &= \int_1^5 \frac{\ln(y)}{y} \int_1^3 \frac{1}{x}\, dx\, dy \\
&= \int_1^5 \frac{\ln(y)}{y} \left. \ln(|x|)\right]_{x=1}^{x=3} dy \\
&= \ln(3) \int_1^5 \frac{\ln(y)}{y}\, dy.
\end{aligned}$$

For the integral $\int_1^5 \frac{\ln(y)}{y}\, dy$ we can use a u-substitution:

$$u = \ln(y) \quad \Rightarrow \quad du = \frac{1}{y}\, dy,$$

and when

$$y = 1 \quad \Rightarrow \quad u = \ln(1) = 0 \qquad \text{and} \qquad y = 5 \quad \Rightarrow \quad u = \ln(5).$$

Then

$$\ln(3) \int_1^5 \frac{\ln(y)}{y}\, dy = \ln(3) \int_0^{\ln(5)} u\ du = \ln(3) \left. \frac{u^2}{2} \right]_0^{\ln(5)} = \frac{\ln(3)\ln(5)^2}{2}.$$

Therefore,

$$\iint_R \frac{\ln(y)}{xy}\,dA = \frac{\ln(3)\ln(5)^2}{2}.$$

Problem 1(b)

$$\int_0^1 \boxed{\int_0^1 \frac{x}{1+xy}\,dy}\,dx$$

u-sub: $u = 1+xy$, $du = x\,dy$

When $y=0 \Rightarrow u=1$, $y=1 \Rightarrow u = 1+x$

$$= \int_0^1 \boxed{\int_1^{1+x} \frac{1}{u}\,du}\,dx$$

$0 \le x \le 1$

$1 \le 1+x \le 2$

$$= \int_0^1 \ln(|u|)\Big]_1^{1+x} dx = \int_0^1 \Big(\ln(|1+x|) - \underbrace{\ln(|1|)}_{=0}\Big)\,dx$$

$$= \int_0^1 \ln(1+x)\,dx$$

Use I.B.P.

Let $u = \ln(1+x)$, $dv = dx$

$du = \frac{1}{1+x}$, $v = x$

$$\int_0^1 u\,dv = uv\Big]_0^1 - \int_0^1 v\,du = x\ln(1+x)\Big]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= x\ln(1+x)\Big]_0^1 - \int_0^1 \frac{x}{1+x}\,dx$$

$$= \ln(2) - 0 - \int_0^1 \frac{x}{1+x}\,dx \quad ?$$

* Another u-sub

Let $u = 1+x$

$\Rightarrow du = dx$

$\Rightarrow x = u-1$

When $x=0 \Rightarrow u=1$, $x=1 \Rightarrow u=2$

$$= \ln(2) - \int_1^2 \frac{u-1}{u}\,du$$

$$= \ln(2) - \int_1^2 \left(1 - \frac{1}{u}\right) du$$

$$= \ln(2) - \big(u - \ln(|u|)\big)\Big]_1^2$$

$$= \ln(2) - \Big[\big(2-\ln(2)\big) - \big(1 - \ln(1)\big)\Big]$$

$$= \ln(2) - \big[1 - \ln(2)\big] = \ln(2) - 1 + \ln(2) = \boxed{2\ln(2) - 1}$$

Problem 1(c)

$$\iint_R xy\sqrt{x^2+y^2}\,dA, \quad \text{where } R=[0,1]\times[0,1].$$

$$= \int_0^1 \int_0^1 xy\sqrt{x^2+y^2}\,dx\,dy$$

u-sub

$u = x^2+y^2$

$du = 2x\,dx$

$\frac{1}{2}du = x\,dx$

When

$x=0 \Rightarrow u=y^2$

$x=1 \Rightarrow u=1+y^2$

$$= \int_0^1 y \int_{y^2}^{1+y^2} \sqrt{u}\,\frac{1}{2}du\,dy$$

$$= \frac{1}{2}\int_0^1 y \boxed{\int_{y^2}^{1+y^2} u^{1/2}\,du}\,dy = \frac{1}{2}\int_0^1 y\left(\frac{2}{3}u^{3/2}\right)\Big]_{y^2}^{1+y^2} dy$$

$$= \frac{1}{3}\int_0^1 y\left[(1+y^2)^{3/2} - (y^2)^{3/2}\right] dy$$

$$= \frac{1}{3}\int_0^1 y(1+y^2)^{3/2} - y^4\,dy$$

$$= \frac{1}{3}\underbrace{\int_0^1 y(1+y^2)^{3/2}dy}_{} - \frac{1}{3}\int_0^1 y^4\,dy$$

Need u-sub

$u = 1+y^2$

$du = 2y\,dy$

$\frac{1}{2}du = y\,dy$

when

$y=0 \Rightarrow u=1$

$y=1 \Rightarrow u=2$

$$= \frac{1}{3}\int_1^2 u^{3/2}\left(\frac{1}{2}du\right) - \frac{1}{3}\int_0^1 y^4\,dy$$

$$= \frac{1}{\cancel{6}_{3}}\cdot\frac{\cancel{2}}{5}u^{5/2}\Big]_1^2 - \frac{1}{3}\cdot\frac{1}{5}y^5\Big]_0^1$$

$$= \frac{1}{15}\left[2^{5/2} - 1^{5/2}\right] - \frac{1}{15} = \frac{2^{5/2}-1-1}{15} = \boxed{\frac{2^{5/2}-2}{15}}$$

Problem 1(d)

$$\int_{-2}^{-1}\int_{0}^{2}\int_{0}^{1} \frac{x^2 e^y}{z}\, dx\, dy\, dz$$

treat y and z as constants

$$= \int_{-2}^{-1}\int_{0}^{2} \frac{e^y}{z} \int_{0}^{1} x^2\, dx\, dy\, dz$$

$$= \int_{-2}^{-1}\int_{0}^{2} \frac{e^y}{z} \left.\frac{x^3}{3}\right]_0^1 dy\, dz = \frac{1}{3}\int_{-2}^{-1}\int_{0}^{2} \frac{e^y}{z}\, dy\, dz$$

$$= \frac{1}{3}\int_{-2}^{-1} \frac{1}{z} \int_{0}^{2} e^y\, dy\, dz$$

$$= \frac{1}{3}\int_{-2}^{-1} \frac{1}{z} \left. e^y\right]_0^2 dz$$

$$= \frac{1}{3}(e^2-1) \int_{-2}^{-1} \frac{1}{z}\, dz$$

$$= \frac{1}{3}(e^2-1) \left.\ln(|z|)\right]_{-2}^{-1}$$

$$= \frac{1}{3}(e^2-1)\left(\underbrace{\ln(|-1|)}_{=0} - \ln(|-2|)\right)$$

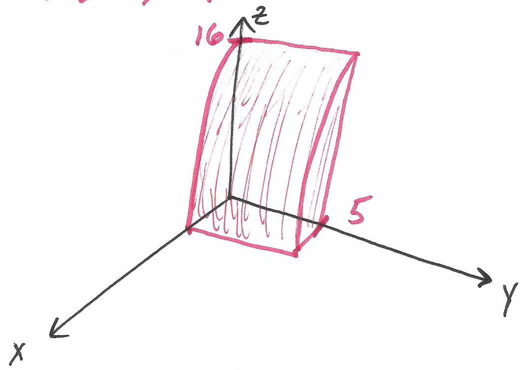
$$= \boxed{-\frac{(e^2-1)\ln(2)}{3}}$$

Problem 5. Find the volume of the solid in the first octant bounded by the parabolic cylinder $z = 16 - x^2$ and the plane $y = 5$.

Find the volume of the solid in the first octant bounded by the parabolic cylinder $z=16-x^2$ and the plane $y=5$.

$x \geq 0, \quad y \geq 0, \quad z \geq 0$

The trace at any plane $y=b$ looks like downward parabola shifted up 16 units



$0 \leq y \leq 5$

$0 \leq x \leq ?$

Determine where $z=16-x^2$ intersects xy-plane ($z=0$)

$0 = 16 - x^2$

$\Rightarrow x = 4, \cancel{-4}$

$0 \leq x \leq 4$

$$\text{Volume of Solid} = \int_0^5 \int_0^4 16 - x^2 \, dx \, dy$$

$$= \int_0^5 16x - \frac{x^3}{3} \Big]_0^4 \, dy$$

$$= \int_0^5 16 \cdot 4 - \frac{4^3}{3} \, dy$$

$$= \left(64 - \frac{64}{3}\right) \int_0^5 1 \, dy$$

$$= \frac{3 \cdot 64 - 64}{3} \quad y \Big]_0^5 = \frac{2 \cdot 64 \cdot 5}{3} = \boxed{\frac{640}{3}}$$