Section 7.8: Trigonometric Integrals

TYPE 2 Improper Integrals are of the form

$$\int_a^b f(x)\,\mathrm{dx},$$

where the function f has any kind of discontinuity over the finite interval $[\mathbf{a}, \mathbf{b}]$.

Problem 2. Determine whether the integral is convergent or divergent. Evaluate the integrals thst are convergent.

(a) $\displaystyle\int_0^{\pi/2} \frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\,\mathrm{d}\theta,$ (b) $\displaystyle\int_0^1 r\ln(r)\,\mathrm{dr}.$

(a) Since $\sqrt{\sin(\pi/2)} = 0$, the function $f(\theta) = \cos(\theta)/\sqrt{\sin(\theta)}$ has a discontinuity at $\theta = \pi/2$. Then we express

$$\int_0^{\pi/2} \frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\,\mathrm{d}\theta = \lim_{t\to(\pi/2)^-} \int_0^t \frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\,\mathrm{d}\theta$$

Let us find $\int_0^t \frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\,\mathrm{d}\theta$. Since the function inside the integral does not appear to be a familiar derivative of some antiderivative, we will try u-substitution. Let

$$u = \sin(\theta) \qquad \Rightarrow \qquad \mathrm{du} = \cos(\theta)\,\mathrm{d}\theta.$$

Substituting the limits of integration, we have

$$\theta = t \quad \Rightarrow u = \cos(t) \quad \text{and} \quad \theta = 0 \quad \Rightarrow u = \cos(0) = 1.$$

Then our definite integral becomes

$$\int_0^t \frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\,\mathrm{d}\theta = \int_1^{\cos(t)} u^{-1/2}\,\mathrm{du} = 2u^{1/2}\Big]_1^{\cos(t)} = 2\left(\sqrt{\cos(t)} - 1\right).$$

Then

$$\int_0^{\pi/2} \frac{\cos(\theta)}{\sqrt{\sin(\theta)}}\,\mathrm{d}\theta = \lim_{t\to(\pi/2)^-} 2\left(\sqrt{\cos(t)} - 1\right) = 2\left(\sqrt{\cos(\pi/2)} - 1\right) = 1(0-1) = -2,$$

since $f(t) = \cos(t)$ is continuous everywhere (meaning we can directly plug in $t = \pi/2$).

(b) Since $y = \ln(r)$ has a vertical asymptote at $r = 0$, i.e., an infinite discontinuity at $r = 0$, then we write

$$\int_0^1 r\ln(r)\,\mathrm{dr} = \lim_{t\to 0^+} \int_t^1 \ln(r)\,\mathrm{dr}.$$

Let us find $\int_t^1 \ln(r)\,\mathrm{dr}$. Note that the function $f(r) = r\ln(r)$ does not appear to be a familiar derivative of some antiderivative, and also, a u-substitution will not help us compute the definite integral, since the derivative $\frac{d}{dr}\ln(r) = 1/r$ does not appear in the integral. **Therefore, we will solve the integral using integration by parts.** Since $y = \ln(r)$ does not have a familiar antiderivative, let

$$u = \ln(r) \quad \Rightarrow \quad \mathrm{du} = \frac{1}{r}\,\mathrm{dr} \quad \text{and} \quad \mathrm{dv} = r \quad \Rightarrow \quad v = \int r\,\mathrm{dr} = \frac{1}{2}r^2.$$

Then

$$\begin{aligned}\int_t^1 \ln(r)\,\mathrm{dr} = \int_t^1 u\,\mathrm{dv} = uv\Big]_t^1 - \int_t^1 v\,\mathrm{du} &= \frac{1}{2}r^2\ln(r)\Big]_t^1 - \int_t^1 \frac{1}{2}r^2\frac{1}{r}\,\mathrm{dr} \\ &= \frac{1}{2}r^2\ln(r)\Big]_t^1 - \frac{1}{2}\int_t^1 r\,\mathrm{dr} \\ &= \frac{1}{2}r^2\ln(r)\Big]_t^1 - \frac{1}{4}r^2\Big]_t^1 \\ &= \frac{1}{2}t^2\ln(t) - \frac{1}{2}1^2\ln(1) - \left(\frac{1}{4}1^2 - \frac{1}{4}t^2\right) \\ &= \frac{1}{2}t^2\ln(t) - \frac{1}{4} + \frac{1}{4}t^2.\end{aligned}$$

Then

$$\begin{aligned}\int_0^1 r\ln(r)\,\mathrm{dr} &= \lim_{t\to 0^+}\left(\frac{1}{2}t^2\ln(t) - \frac{1}{4} + \frac{1}{4}t^2\right) \\ &= \frac{1}{2}\lim_{t\to 0^+} t^2\ln(t) - \lim_{t\to 0^+}\frac{1}{4} + \frac{1}{4}\lim_{t\to 0^+} t^2.\end{aligned}$$

Note that $\lim_{t\to 0^+}\frac{1}{4} = \frac{1}{4}$ and $\lim_{t\to 0^+} t^2 = 0^2 = 0$. However,

$$\text{as} \quad t \to 0^+ \quad \Rightarrow t^2 \to 0 \quad \text{and} \quad \ln(t) \to -\infty,$$

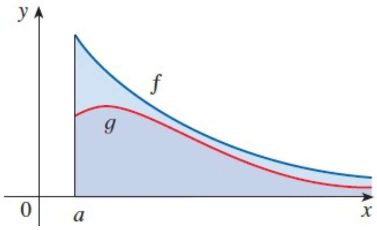
meaning that the limit $\lim_{\mathbf{t}\to \mathbf{0}^+} \mathbf{t}^2\ln(\mathbf{t})$ **is in** $0\cdot\infty$ **indeterminate form.** Therefore, we can apply L'Hospital's Rule to the limit. We have

$$\lim_{t\to 0^+} t^2\ln(t) = \lim_{t\to 0^+}\frac{\ln(t)}{t^{-2}} = \lim_{t\to 0^+}\frac{t^{-1}}{-2t^{-3}} = \lim_{t\to 0^+} -\frac{1}{2}t^2 = -\frac{1}{2}0^2 = 0.$$

Then

$$\int_0^1 r\ln(r)\,\mathrm{dr} = \frac{1}{2}\lim_{t\to 0^+} t^2\ln(t) - \lim_{t\to 0^+}\frac{1}{4} + \frac{1}{4}\lim_{t\to 0^+} t^2 = \frac{1}{2}\cdot 0 - \frac{1}{4} - \frac{1}{4}\cdot 0 = -\frac{1}{4}.$$

Using the Comparison Theorem



Comparison Theorem Suppose that f and g are continuous functions with $f(x) \geqslant g(x) \geqslant 0$ for $x \geqslant a$.
(a) If $\int_a^\infty f(x)\,dx$ is convergent, then $\int_a^\infty g(x)\,dx$ is convergent.
(b) If $\int_a^\infty g(x)\,dx$ is divergent, then $\int_a^\infty f(x)\,dx$ is divergent.

Problem 3. Use the Comparison Theorem to determine whether the integral is convergent or divergent. **You do not have to evaluate the integral.**

(a) $\displaystyle\int_1^\infty \frac{1+\sin^2(x)}{\sqrt{x}}\,\mathrm{dx}$, (b) $\displaystyle\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx}$ **Hint:** "Split" the integral at $x = 2$.

Recall the following fact, learned in class:

$$\int_1^\infty \frac{1}{x^p}\,dx \quad \text{is convergent if } p > 1 \text{ and divergent if } p \leqslant 1.$$

Note: We often use the function $f(x) = \frac{1}{x^p}$ to compare with another function f to show that $\int_a^\infty f(x)\,\mathrm{dx}$ converges or diverges using the Comparison Theorem. More specifically:

$$\text{If } \frac{1}{x^p} \geq f(x) \text{ and } p > 1 \text{ then } \int_a^\infty f(x)\,\mathrm{dx} \text{ converges}$$

AND

$$\text{if } \frac{1}{x^p} \leq f(x) \text{ and } p \leq 1 \text{ then } \int_a^\infty f(x)\,\mathrm{dx} \text{ diverges.}$$

(a) Note that since $\sin^2(x) \geq 0$, then

$$\frac{1+\sin^2(x)}{\sqrt{x}} \geq \frac{1+0}{\sqrt{x}} = \frac{1}{x^{1/2}}$$

for all $x \geq 1$. Then since $\displaystyle\int_1^\infty \frac{1}{x^{1/2}}\,\mathrm{dx}$ is divergent, by the Comparison Theorem, $\displaystyle\int_1^\infty \frac{1+\sin^2(x)}{\sqrt{x}}\,\mathrm{dx}$ is also divergent.

(b) The function $f(x) = \frac{x+1}{\sqrt{x^4-x}}$ has a discontinuity at $x = 1$, since $f(1)$ is undefined. Then if we express

$$\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx} = \int_1^2 \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx} + \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx},$$

note that $I_1 = \int_1^2 \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx}$ is a Type 2 integral and $I_2 = \int_2^\infty \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx}$ is a Type 1 integral. We will show that I_2 is divergent using the Comparison Theorem, which will imply that the original integral $\int_1^\infty \frac{1}{x^{1/2}}\,\mathrm{dx}$ is divergent.

Note that

$$\frac{x+1}{\sqrt{x^4-x}} \geq \frac{x+1}{\sqrt{x^4}} \geq \frac{x}{\sqrt{x^4}} = \frac{x}{(x^4)^{1/2}} = \frac{x}{x^2} = \frac{1}{x}.$$

We claim that $\int_2^\infty \frac{1}{x}\,dx$ is divergent. Suppose this is **not** the case; that is, suppose that $\int_2^\infty \frac{1}{x}$ is convergent. Then since

$$\int_1^\infty \frac{1}{x}\,\mathrm{dx} = \int_1^2 \frac{1}{x}\,\mathrm{dx} + \int_2^\infty \frac{1}{x}\,\mathrm{dx},$$

and the integrals $\int_1^2 \frac{1}{x}\,\mathrm{dx}$ and $\int_2^\infty \frac{1}{x}\,\mathrm{dx}$ are convergent (the former is because it is not improper and is a definite integral), this implies that $\int_1^\infty \frac{1}{x}\,\mathrm{dx}$, which is a contradiction! This contradicts the fact that $\int_1^\infty \frac{1}{x}\,\mathrm{dx}$ is divergent! Therefore, $\int_2^\infty \frac{1}{x}\,dx$ is divergent.

Therefore, by the Comparison Theorem, I_2 is divergent, and hence, $\int_1^\infty \frac{x+1}{\sqrt{x^4-x}}\,\mathrm{dx} = I_1 + I_2$ is divergent.