Section 14.2: Limits & Continuity of 2-Variable Functions

Problem 1. Show that the following limits **do not exist**.

(a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4}$ (b) $\displaystyle\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)}$ (c) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy+yz^2+xz^2}{x^2+y^2+z^2}$

(a) Let

$$f(x,y) = \frac{y^2\sin^2(x)}{x^4+y^4}.$$

Let C_1 be the path where the point (x,y) approaches the point $(0,0)$ along the line $x=0$ (the y-axis). Then since

$$f(0,y) = \frac{y^2\sin^2(0)}{0^4+y^4} = \frac{0}{y^4},$$

we have

$$\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4} = \lim_{(x,y)\to(0,0)} \frac{0}{y^4} = 0,$$

since $y \to 0$, but $y \neq 0$.

NOTE: The path $y=0$ (the x-axis) will also lead us to $f(x,y)\to 0$, so we will need to consider another path that passes through $(0,0)$.

Let C_2 be the path where the point (x,y) approaches the point $(1,1)$ along the line $y=x$. Then since

$$f(x,x) = \frac{x^2\sin^2(x)}{2x^4} = \frac{\sin^2(x)}{2x^2},$$

we have

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{y^2\sin^2(x)}{x^4+y^4} &= \lim_{(x,y)\to(0,0)} \frac{\sin^2(x)}{2x^2} \\
&\overset{\text{L'Hop}}{=} \lim_{(x,y)\to(0,0)} \frac{2\sin(x)\cos(x)}{4x} \\
&\overset{\text{L'Hop}}{=} \lim_{(x,y)\to(0,0)} \frac{2(\cos^2(x)-\sin^2(x))}{4} \\
&= \frac{2(1-0)}{4} \\
&= \frac{1}{2}.
\end{aligned}$$

NOTE: We can also compute the limit $\lim_{(x,y)\to(0,0)} \frac{\sin^2(x)}{2x^2}$ in the following way:

$$\begin{aligned}
\lim_{(x,y)\to(0,0)} \frac{\sin^2(x)}{2x^2} \\
&= \frac{1}{2} \lim_{(x,y)\to(0,0)} \left(\frac{\sin(x)}{x}\right)^2 \\
&= \frac{1}{2} \left(\lim_{(x,y)\to(0,0)} \frac{\sin(x)}{x}\right)^2 \\
&\overset{\text{L'Hop}}{=} \frac{1}{2} \left(\lim_{(x,y)\to(0,0)} \frac{\cos(x)}{1}\right)^2 \\
&= \frac{1}{2}1^2 \\
&= \frac{1}{2}.
\end{aligned}$$

Since $f(x,y) \to 0$ as $(x,y) \to (0,0)$ along the path C_1 and $f(x,y) \to 1/2$ as $(x,y) \to (0,0)$ along the path C_2, we have shown that $\lim_{(x,y)\to(0,0)} \frac{y^2 \sin^2(x)}{x^4+y^4}$ DNE.

(b) **NOTE:** Make sure that the path you choose passes through the point that (x,y) is approaching in the limit. For example, for this limit, we cannot choose the line $x = 0$ (the y-axis) because it does not pass through the point $(1,1)$.

Let

$$g(x,y) = \frac{y-x}{1-y+\ln(x)}.$$

Let C_1 be the path where the point (x,y) approaches the point $(1,1)$ along the line $x = 1$. Then since

$$g(1,y) = \frac{y-1}{1-y+\ln(1)} = \frac{y-1}{1-y},$$

we have

$$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)} = \lim_{y\to1} \frac{y-1}{1-y} = \lim_{y\to1} \frac{y-1}{-(y-1)} = \lim_{y\to1} -1 = -1,$$

since $y \to 1$, but $y \neq 1$.

Let C_2 be the path where the point (x,y) approaches the point $(1,1)$ along the line $y = x$. Then

$$g(x,x) = \frac{x-x}{1-x+\ln(x)} = \frac{0}{1-x+\ln(x)} = 0,$$

since $1 - x + \ln(x) \to 0$ as $x \to 1$, but $1 - x + \ln(x) \neq 0$. Then

$$\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)} = \lim_{(x,y)\to(1,1)} 0 = 0.$$

Since $g(x,y) \to -1$ as $(x,y) \to (1,1)$ along the path C_1 and $g(x,y) \to 0$ as $(x,y) \to (1,1)$ along the path C_2, we have shown that $\displaystyle\lim_{(x,y)\to(1,1)} \frac{y-x}{1-y+\ln(x)}$ DNE.

(c) You can use a similar approach for finding limits of 3-variable functions than you would with 2-variable functions. Let

$$h(x,y,z) = \frac{xy + yz^2 + xz^2}{x^2+y^2+z^2}.$$

Let C_1 be the path where the point (x,y,z) approaches $(0,0,0)$ along the x-axis. Then $y = z = 0$ and we have

$$h(x,0,0) = \frac{0}{x^2}.$$

Then

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2+y^2+z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{0}{x^2} = 0,$$

since $x \to 0$, but $x \neq 0$.

Let C_2 be the path where the point (x,y,z) approaches $(0,0,0)$ along the path $y = x$, $z = 0$ (this is the line $y = x$ on the xy-plane, so it passes through the origin). Then we have

$$h(x,x,0) = \frac{x^2}{2x^2}.$$

Then

$$\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2+y^2+z^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{x^2}{2x^2} = \lim_{(x,y,z)\to(0,0,0)} \frac{1}{2} = \frac{1}{2}.$$

Since $h(x,y,z) \to 0$ as $(x,y,z) \to (0,0,0)$ along the path C_1 and $h(x,y,z) \to 1/2$ as $(x,y,z) \to (0,0,0)$ along the path C_2, we have shown that $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{xy + yz^2 + xz^2}{x^2+y^2+z^2}$ DNE.

Problem 2.

(a) Show that $\displaystyle\lim_{(x,y)\to(1,0)} (x-1)^2 \cos\left(\frac{1}{y}\right) = 0.$ (b) Show that $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$

NOTE: Since both of the functions in the limits shown above are undefined at the point which (x,y) is approaching, and we are given that each of the limits exist, we can apply the **Squeeze Theorem** to solve these limits.

(a) We have

$$\left|(x-1)^2 \cos\left(\frac{1}{y}\right)\right| = (x-1)^2 \left|\cos\left(\frac{1}{y}\right)\right| \leq (x-1)^2,$$

where the last inequality holds since $|\cos(A)| \leq 1$, regardless of what A is. Then since

$$\left|(x-1)^2 \cos\left(\frac{1}{y}\right)\right| \leq (x-1)^2 \quad \Leftrightarrow \quad -(x-1)^2 \leq (x-1)^2 \cos\left(\frac{1}{y}\right) \leq (x-1)^2,$$

We have found a way to "squeeze" the function $F(x,y) = (x-1)^2 \cos\left(\frac{1}{y}\right)$ between two functions. Note that

$$\lim_{(x,y)\to(1,0)} (x-1)^2 = 0,$$

and similarly, $\lim_{(x,y)\to(1,1)} -(x-1)^2 = 0$. Therefore, **by the Squeeze Theorem,**

$$\lim_{(x,y)\to(1,0)} (x-1)^2 \cos\left(\frac{1}{y}\right) = 0.$$

(b) We have

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| = \frac{|xy|}{\sqrt{x^2+y^2}} \leq |xy|.$$

Then since

$$\left| \frac{xy}{\sqrt{x^2+y^2}} \right| \leq |xy| \quad \Leftrightarrow \quad -|xy| \leq \frac{xy}{\sqrt{x^2+y^2}} \leq |xy|,$$

We have found a way to "squeeze" the function $G(x,y) = \frac{xy}{\sqrt{x^2+y^2}}$ between two functions. Note that

$$\lim_{(x,y)\to(0,0)} |xy| = 0,$$

and similarly, $\lim_{(x,y)\to(0,0)} -|xy| = 0$. Therefore, **by the Squeeze Theorem,**

$$\lim_{(x,y)\to(0,0)} \frac{xy}{\sqrt{x^2+y^2}} = 0.$$

Problem 3. Determine the set of points at which the function is continuous.

(a) $F(x,y) = \dfrac{xy}{1+e^{x-y}}$.

(b) $g(x,y) = \dfrac{e^x+e^y}{e^{xy}-1}$

(c) $f(x,y) = \begin{cases} \dfrac{xy}{x^2+xy+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0). \end{cases}$

(a) Note that $F(x,y) = \dfrac{xy}{1+e^{x-y}}$ consists of the quotient of a polynomial, $f(x,y) = xy$, and an exponential function, $g(x,y) = 1+e^{x-y}$, which are both continuous on $\mathbb{R}^2$ (polynomials and exponential functions are continuous everywhere). Since $e^{x-y} > 0$ for **any** values of x and y, then $1+e^{x-y} \neq 0$ for all values of x and y. Therefore, F has no discontinuities, i.e., F is continuous on $\mathbb{R}^2$.

(b) Note that $g(x, y) = \dfrac{e^x + e^y}{e^{xy} - 1}$ consists of the quotient of two exponential functions, which are both continuous on $\mathbb{R}^2$ (exponential functions are continuous everywhere). Since

$$e^{xy} - 1 = 0 \qquad \Leftrightarrow \qquad e^{xy} = 1 \qquad \Leftrightarrow \qquad xy = \ln(1) = 0,$$

then g has a discontinuities at any point (x, y) that satisfies $xy = 0$, which is only when $x = 0$ or $y = 0$ (this also includes the case when both are equal to 0). Therefore, g is continuous on the set

$$\{(xy) \mid x \neq 0 \text{ AND } y \neq 0\}.$$

Note that the set above consists of all points that are neither on the x-axis nor the y-axis on the xy-plane.

(c) Let

$$h(x, y) = \frac{xy}{x^2 + xy + y^2}.$$

Note that h is a rational function and hence, is continuous everywhere on its domain. Note that

$$\text{dom}(h) = \left\{(x, y) \mid x^2 + xy + y^2 \neq 0\right\} = \{(x, y) \mid (x, y) \neq (0, 0)\}.$$

Therefore, h is continuous everywhere except at the point $(0, 0)$. Since $f(x, y) = \begin{cases} h(x, y) & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0), \end{cases}$ then f itself may be continuous everywhere, except possibly at $(0, 0)$. We will show that this is indeed the case by showing that $\lim_{(x,y)\to(0,0)} f(x, y)$ DNE.

Let C_1 be the path where the point (x, y) approaches the point $(0, 0)$ along the line $y = 0$. Since

$$f(x, 0) = \frac{0}{x^2} = 0,$$

we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} 0 = 0.$$

Let C_2 be the path where the point (x, y) approaches the point $(0, 0)$ along the line $y = x$. Since

$$f(x, x) = \frac{x^2}{x^2 + x^2 + x^2} = \frac{x^2}{3x^2} = \frac{1}{3},$$

since $x \to 0$, but $x \neq 0$, we have

$$\lim_{(x,y)\to(0,0)} \frac{xy}{x^2 + xy + y^2} = \lim_{x\to 0} \frac{1}{3} = \frac{1}{3}.$$

Since f has two different limits along two different lines, **the limit DNE as $(\mathbf{x}, \mathbf{y}) \to \mathbf{0}$**, which implies that f is **not continuous at $(\mathbf{0}, \mathbf{0})$, but is continuous at every other point on the xy-plane**.

Section 14.3: Partial Derivatives

Problem 4. Find the partial derivatives of the following functions:

(a) $f(x, y) = y(x + x^2y)^5$

(b) $u(r, \theta) = \sin\left(\dfrac{r\cos(\theta)}{\theta^2}\right)$

(c) $w = xy^2e^{-xz}$

(a)

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\left(y(x+x^2y)^5\right) = y\frac{\partial}{\partial x}\left((x+x^2y)^5\right) = 5y(x+x^2y)^4\frac{\partial}{\partial x}\left((x+x^2y)\right) = 5y(x+x^2y)^4(1+2xy)$$

$$\begin{aligned} f_y = \frac{\partial f}{\partial y} = \frac{\partial}{\partial y}\left(y(x+x^2y)^5\right) &= \left(\frac{\partial}{\partial y}y\right)(x+x^2y)^5 + y\left(\frac{\partial}{\partial y}(x+x^2y)^5\right) \\ &= (x+x^2y)^5 + 5y(x+x^2y)^4\frac{\partial}{\partial y}(x+x^2y) \\ &= (x+x^2y)^5 + 5y(x+x^2y)^4x^2 \end{aligned}$$

(b)

$$u_r = \frac{\partial u}{\partial r} = \cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\frac{\partial}{\partial r}\left(\frac{r\cos(\theta)}{\theta^2}\right) = \cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\left(\frac{\cos(\theta)}{\theta^2}\right)$$

$$\begin{aligned} u_\theta = \frac{\partial u}{\partial \theta} = \cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\frac{\partial}{\partial \theta}\left(\frac{r\cos(\theta)}{\theta^2}\right) &= r\cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\frac{\partial}{\partial \theta}\left(\frac{\cos(\theta)}{\theta^2}\right) \\ &= r\cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\left(\frac{\left(\frac{\partial}{\partial\theta}\cos(\theta)\right)\theta^2 - \cos(\theta)\left(\frac{\partial}{\partial\theta}\theta^2\right)}{\theta^4}\right) \\ &= r\cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\left(\frac{\left(\frac{\partial}{\partial\theta}\cos(\theta)\right)\theta^2 - \cos(\theta)\left(\frac{\partial}{\partial\theta}\theta^2\right)}{\theta^4}\right) \\ &= r\cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\left(\frac{-\sin(\theta)\theta^2 - 2\cos(\theta)\theta}{\theta^4}\right) \\ &= r\cos\left(\frac{r\cos(\theta)}{\theta^2}\right)\left(\frac{-\sin(\theta)\theta - 2\cos(\theta)}{\theta^3}\right) \end{aligned}$$

(c)

$$w_x = \frac{\partial w}{\partial x} = \frac{\partial}{\partial x}\left(xy^2e^{-xz}\right) = y^2\frac{\partial}{\partial x}\left(xe^{-xz}\right) = y^2\left(\left(\frac{\partial}{\partial x}x\right)e^{-xz} + x\left(\frac{\partial}{\partial x}e^{-xz}\right)\right) = y^2\left(e^{-xz} - xze^{-xz}\right)$$

$$w_y = \frac{\partial w}{\partial y} = \frac{\partial}{\partial y}\left(xy^2e^{-xz}\right) = xe^{-xz}\frac{\partial}{\partial y}y^2 = 2xe^{-xz}y$$

$$w_z = \frac{\partial w}{\partial z} = \frac{\partial}{\partial z}\left(xy^2e^{-xz}\right) = xy^2\frac{\partial}{\partial z}e^{-xz} = -x^2y^2e^{-xz}$$

Problem 5. Let $f(x,y,z) = x^{yz}$. Find $\dfrac{\partial^2 f}{\partial z \partial x}$ at the point $(e,1,0)$.

HINT: Remember the following differentiation formula from Calculus I:

$$\tfrac{d}{dt}b^t = \ln(b)b^t \text{ for any constant } b > 0$$

We have

$$f_x = \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} x^{yz} = yz\, x^{yz-1}.$$

Then

$$\begin{aligned}
&\frac{\partial^2 f}{\partial z \partial x} \\
&= \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\right) \\
&= \frac{\partial}{\partial z}\left(yz\, x^{yz-1}\right) \\
&= \left(\frac{\partial}{\partial z} yz\right) x^{yz-1} + yz\left(\frac{\partial}{\partial z} x^{yz-1}\right) \\
&= \left(\frac{\partial}{\partial z} yz\right) x^{yz-1} + yz\mathbf{x^{-1}}\left(\frac{\partial}{\partial z} x^{yz}\right) \quad \text{(factor out } x^{-1} \text{ from } x^{yz-1} \text{ to apply the given differentiation formula)} \\
&= yx^{yz-1} + \frac{yz}{x}\ln(x) x^{yz} \frac{\partial}{\partial z} yz \\
&= yx^{yz-1} + \frac{yz}{x}\ln(x) x^{yz} y \\
&= yx^{yz-1} + \frac{y^2 z}{x}\ln(x) x^{yz}.
\end{aligned}$$

Evaluating this second order partial derivative at the point $(e, 1, 0)$ gives us

$$\left.\frac{\partial^2 f}{\partial z \partial x}\right|_{x=e, y=1, z=0} = 1 \cdot e^{1(0)-1} + \frac{1^2(0)}{e}\ln(e) e^{1(0)} = e^{-1} = \frac{1}{e}.$$

Problem 6. Find all the second partial derivatives of $z = \frac{y}{2x+3y}$.

In class, we found that $\frac{\partial z}{\partial x} = -\frac{2y}{(2x+3y)^2}$ and $\frac{\partial z}{\partial y} = \frac{2x}{(2x+3y)^2}$.

$$z_{xx} = \frac{\partial^2 z}{\partial x^2} = \frac{\partial}{\partial x}\left(-\frac{2y}{(2x+3y)^2}\right) = -2y\frac{\partial}{\partial x}\left((2x+3y)^{-2}\right) = \frac{8y}{(2x+3y)^3}$$

$$\begin{aligned}
z_{xy} = \frac{\partial^2 z}{\partial y \partial x} = \frac{\partial}{\partial y}\left(-\frac{2y}{(2x+3y)^2}\right) &= -\left(\frac{2(2x+3y)^2 - 2y(2)(2x+3y)3}{(2x+3y)^4}\right) \\
&= \frac{-2(2x+3y) + 12y}{(2x+3y)^3} \\
&= \frac{-4x + 6y}{(2x+3y)^3}
\end{aligned}$$

By Clairaut's Theorem, we have

$$z_{yx} = z_{xy} = \frac{-4x + 6y}{(2x+3y)^3}.$$

$$z_{yy} = \frac{\partial^2 z}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{2x}{(2x+3y)^2}\right) = 2x\frac{\partial}{\partial y}(2x+3y)^{-2} = \frac{-12x}{(2x+3y)^3}$$

Problem 7. Find the indicated partial derivative. **HINT:** You can change the order of differentiation using Clairaut's Theorem.
$w = \sqrt{u + v^2} + u\sin^2(t)$; $\qquad \dfrac{\partial^4 w}{\partial v^2 \partial t \partial u}$.

We can rewrite the function as

$$w = (u + v^2)^{1/2} + u\sin^2(t).$$

Notice that the left term of the function does not involve t and the right side of the function does not involve v. As a result, we can easily obtain the order 4 derivative, $\frac{\partial^4 w}{\partial v^2 \partial t \partial u}$, by changing the order of differentiation using Clairaut's Theorem and taking the derivatives with respect to t and v first.

By Clairaut's Theorem, we have

$$\frac{\partial^4 w}{\partial v^2 \partial t \partial u} = w_{utvv} = w_{tvvu} = \frac{\partial^4 w}{\partial u \partial v^2 \partial t}.$$

Then

$$w_t = \frac{\partial w}{\partial t} = \frac{\partial}{\partial t}\left((u+v^2)^{1/2} + u\sin^2(t)\right) = 0 + u\frac{\partial}{\partial t}\sin^2(t) = 2u\,\sin(t)\cos(t)$$

$$\Downarrow$$

$$w_{tv} = \frac{\partial^2 w}{\partial v \partial t} = \frac{\partial}{\partial v}2u\,\sin(t)\cos(t) = 0$$

$$\Downarrow$$

$$w_{tvv} = \frac{\partial^3 w}{\partial v^2 \partial t} = \frac{\partial}{\partial v}0 = 0$$

$$\Downarrow$$

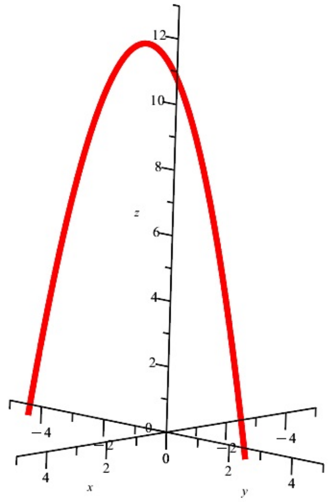
$$w_{tvvu} = \frac{\partial^4 w}{\partial u \partial v^2 \partial t} = \frac{\partial}{\partial u}0 = 0.$$

Problem 8. Let $f(x, y) = 16 - 4x^2 - y^2$.
(a) Use trace curves to determine the graph of f.
(b) Find $f_x(1, 2)$ and $f_y(1, 2)$ and interpret these numbers as slopes of tangent lines to trace curves.

(a) We will find the traces of the function at the planes $x = 1$, $y = 2$, and $z = 0$. At $x = 1$ we have

$$z = f(1, y) = 16 - 4 - y^2 = -y^2 + 12$$

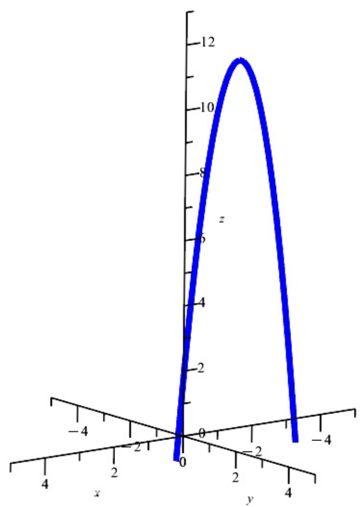
so the trace is a parabola.



At $y = 2$ we have

$$z = f(x, 2) = 16 - 4x^2 - 2^2 = -4x^2 + 12$$

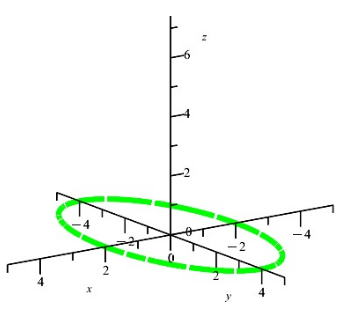
so the trace is the upper half of an ellipse centered at the origin with horizontal radius 2 and vertical radius 4 on the xz-plane.



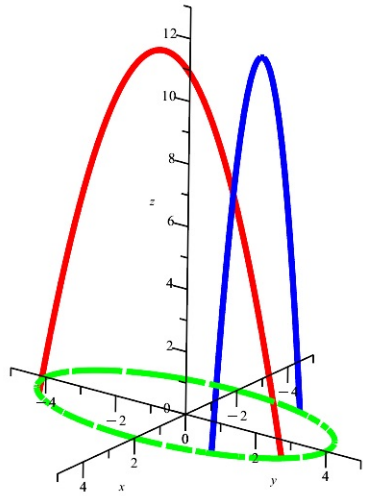
At $z = 0$ we have

$$0 = 16 - 4x^2 - y^2 \quad \Rightarrow \quad \frac{x^2}{2^2} + \frac{y^2}{4^2} = 1,$$

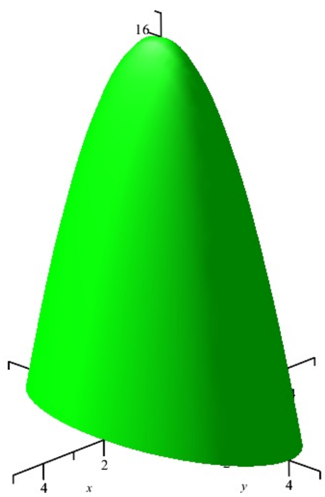
so the trace is an ellipse centered at the origin with horizontal radius 2 and vertical radius 4 on the xy-plane.



By plotting all of the trace curves in 3D space we have



The graph of the function is a paraboloid (vertical traces are parabolas, horizontal traces are ellipses).



(b) We have

$$f_x(x, y) = -8x \qquad \text{and} \qquad f_y(x, y) = -2y.$$

Then

$$f_x(1, 2) = -8 \qquad \text{and} \qquad f_y(1, 2) = -2(2) = -4.$$

The value $f_x(1, 2) = -8$ is the slope of the tangent line to the trace curve $z = f(x, 2) = -4x^2 + 12$ at $x = 1$ and the value $f_y(1, 2) = -4$ is the slope of the tangent line to the trace curve $z = f(1, y) = -y^2 + 12$ at $y = 2$.