MAT 2500 (Dr. Fuentes) Worksheet 5 - Sections 13.3 & 13.4

Section 13.3: Arc Length

Problem 1. Find the length of each of the following curves over the given range of t.

(a) $\mathbf{r}(t) = \left\langle 2t, t^2, \frac{1}{3}t^3 \right\rangle,\ 0 \le t \le 1$, (b) $\mathbf{r}(t) = \mathbf{i} + t^2\mathbf{j} + t^3\mathbf{k},\ 0 \le t \le 1$.

(a) We have

$$\begin{aligned}
\text{Arc Length} = \int_0^1 |\mathbf{r}'(t)|\,\mathrm{dt} = \int_0^1 |\left\langle 2, 2t, t^2 \right\rangle|\,\mathrm{dt} &= \int_0^1 \sqrt{2^2 + (2t)^2 + (t^2)^2}\,\mathrm{dt} \\
&= \int_0^1 \sqrt{4 + 4t^2 + t^4}\,\mathrm{dt} \\
&= \int_0^1 \sqrt{(2 + t^2)^2}\,\mathrm{dt} \\
&= \int_0^1 (2 + t^2)\,\mathrm{dt}, \quad \text{since } 2 + t^2 > 0, \\
&= \left(2t + \frac{1}{3}t^3\right)\Big]_0^1 = 2 + \frac{1}{3} = \frac{7}{3}.
\end{aligned}$$

(b) We have

$$\begin{aligned}
\text{Arc Length} = \int_0^1 |\mathbf{r}'(t)|\,\mathrm{dt} = \int_0^1 |\left\langle 0, 2t, 3t^2 \right\rangle|\,\mathrm{dt} &= \int_0^1 \sqrt{0^2 + (2t)^2 + (3t^2)^2}\,\mathrm{dt} \\
&= \int_0^1 \sqrt{4t^2 + 9t^4}\,\mathrm{dt} \\
&= \int_0^1 t\sqrt{4 + 9t^2}\,\mathrm{dt}, \quad \text{since } 0 \le t \le 1.
\end{aligned}$$

Let $u = 4 + 9t^2$. Then $(1/18)\mathrm{du} = t^2\,\mathrm{dt}$, when $t = 0$ we have $u = 4$ and when $t = 1$ we have $u = 13$. Then

$$\int_0^1 t\sqrt{4 + 9t^2}\,\mathrm{dt} = \frac{1}{18}\int_4^{13} u^{1/2}\,\mathrm{du} = \frac{1}{18}\left(\frac{2}{3}u^{3/2}\right)\Big]_4^{13} = \frac{1}{27}\left(13^{3/2} - 4^{3/2}\right) = \frac{1}{27}\left(13^{3/2} - 8\right).$$

Problem 2. Find the arc length function for the curve measured from the point P in the direction of increasing t and then reparametrize the curve with respect to arc length starting from P.

$$\mathbf{r}(t) = (5 - t)\mathbf{i} + (4t - 3)\mathbf{j} + 3t\mathbf{k}, \qquad P = (4, 1, 3).$$

Since $P = (4, 1, 3)$, then $3t = 3$, meaning that the point P corresponds to $t = 1$. We have $\mathbf{r}'(t) = \left\langle -1, 4, 3 \right\rangle$. Let

$$s = s(t) = \int_1^t |\mathbf{r}'(x)|\,\mathrm{dx}.$$

Then

$$\frac{\mathrm{ds}}{\mathrm{dt}} = |\mathbf{r}'(x)| = \sqrt{1 + 16 + 9} = \sqrt{26}.$$

Then

$$s = s(t) = \int_1^t \sqrt{26}\,\mathrm{d}x = \sqrt{26}x\Big]_1^t = \sqrt{26}(t-1).$$

That is, $s = \sqrt{26}(t-1)$. By solving for t we obtain

$$t = t(s) = \frac{s}{\sqrt{26}} + 1.$$

By substituting $t = t(s)$ into $\mathbf{r}(t)$ we obtain a reparametrization of $\mathbf{r}(t)$ as follows:

$$\mathbf{r}(t(s)) = \mathbf{r}\left(\frac{s}{\sqrt{26}}+1\right) = \left\langle 5 - \left(\frac{s}{\sqrt{26}}+1\right), 4\left(\frac{s}{\sqrt{26}}+1\right) - 3, 3\left(\frac{s}{\sqrt{26}}+1\right)\right\rangle$$
$$= \left\langle 4 - \frac{s}{\sqrt{26}}, \frac{4s}{\sqrt{26}}+1, \frac{3s}{\sqrt{26}}+3\right\rangle.$$

Problem 3. Suppose you start at the point $(0,0,3)$ and move 5 units along the curve $\mathbf{r}(t) = 3\sin(t)\mathbf{i} + 4t\mathbf{j} + 3\cos(t)\mathbf{k}$ in the positive direction. At what point on the curve are you now?

HINT: The answer is not $(5,5,8)$ nor is it $\mathbf{r}(5) = \left\langle 3\sin(5), 32, 3\cos(5)\right\rangle$. You must locate the point at which the arc length equals 5 starting from the point $(0,0,3)$. Solve $\int_0^t |\mathbf{r}'(u)|\,\mathrm{d}u = 5$ for t to find the point.

Here $\mathbf{r}(t) = \langle 3\sin t, 4t, 3\cos t\rangle$, so $\mathbf{r}'(t) = \langle 3\cos t, 4, -3\sin t\rangle$ and $|\mathbf{r}'(t)| = \sqrt{9\cos^2 t + 16 + 9\sin^2 t} = \sqrt{25} = 5$.

The point $(0,0,3)$ corresponds to $t = 0$, so the arc length function beginning at $(0,0,3)$ and measuring in the positive direction is given by $s(t) = \int_0^t |\mathbf{r}'(u)|\,du = \int_0^t 5\,du = 5t$. $s(t) = 5 \Rightarrow 5t = 5 \Rightarrow t = 1$, thus your location after moving 5 units along the curve is $(3\sin 1, 4, 3\cos 1)$.

Section 13.4: Motion in Space: Velocity and Acceleration

Problem 4. Find the position vector of a function that has

$$\text{acceleration vector } \mathbf{a}(t) = 2t\mathbf{i} + \sin(t)\mathbf{j} + \cos(2t)\mathbf{k},$$

$$\text{initial velocity } \mathbf{v}(0) = \mathbf{i}, \text{ and}$$

$$\text{initial position } \mathbf{r}(0) = \mathbf{j}.$$

The velocity vector of the particle is

$$\mathbf{v}(t) = \int \mathbf{a}(t)\,\mathrm{d}t = \left\langle \int 2t\,\mathrm{d}t, \int \sin(t)\,\mathrm{d}t, \int \cos(2t)\,\mathrm{d}t\right\rangle = \left\langle t^2, -\cos(t), \frac{1}{2}\sin(2t)\right\rangle + \mathbf{C}.$$

Given that $\mathbf{v}(0) = \mathbf{i} = \left\langle 1,0,0\right\rangle$, and using our formula for $\mathbf{v}(t)$, we have

$$\left\langle 1,0,0\right\rangle = \mathbf{v}(0) = \left\langle 0^2, -\cos(0), \frac{1}{2}\sin(0)\right\rangle + \mathbf{C} \Rightarrow \left\langle 1,0,0\right\rangle = \left\langle 0, -1, 0\right\rangle + \mathbf{C}$$
$$\Rightarrow \mathbf{C} = \left\langle 1,0,0\right\rangle - \left\langle 0,-1,0\right\rangle = \left\langle 1,1,0\right\rangle.$$

Then

$$\mathbf{v}(t) = \left\langle t^2,\, -\cos(t),\, \frac{1}{2}\sin(2t)\right\rangle + \mathbf{C} = \left\langle t^2,\, -\cos(t),\, \frac{1}{2}\sin(2t)\right\rangle + \left\langle 1,1,0\right\rangle = \left\langle t^2+1,\, -\cos(t)+1,\, \frac{1}{2}\sin(2t)\right\rangle.$$

The position vector of the particle is

$$\begin{aligned}\mathbf{r}(t) = \int \mathbf{v}(t)\,\mathrm{dt} &= \left\langle \int t^2+1\,\mathrm{dt},\, \int -\cos(t)+1\,\mathrm{dt},\, \int \frac{1}{2}\sin(2t)\,\mathrm{dt}\right\rangle \\ &= \left\langle \frac{1}{3}t^3+t,\, -\sin(t)+t,\, -\frac{1}{4}\cos(2t)\right\rangle + \mathbf{C}^*.\end{aligned}$$

Given that $\mathbf{r}(0) = \mathbf{j} = \left\langle 0,1,0\right\rangle$, and using our formula for $\mathbf{r}(t)$, we have

$$\begin{aligned}\left\langle 0,1,0\right\rangle = \mathbf{r}(0) = \left\langle \frac{1}{3}0^3+0,\, -\sin(0),\, -\frac{1}{4}\cos(0)\right\rangle + \mathbf{C}^* &\Rightarrow \left\langle 0,1,0\right\rangle = \left\langle 0,\, 0,\, -1/4\right\rangle + \mathbf{C}^* \\ &\Rightarrow \mathbf{C}^* = \left\langle 0,1,0\right\rangle - \left\langle 0,\, 0,\, -1/4\right\rangle = \left\langle 0,1,1/4\right\rangle.\end{aligned}$$

Then the position vector of the particle is

$$\begin{aligned}\mathbf{r}(t) = \left\langle \frac{1}{3}t^3+t,\, -\sin(t),\, -\frac{1}{4}\cos(2t)\right\rangle + \mathbf{C}^* &= \left\langle \frac{1}{3}t^3+t,\, -\sin(t)+t,\, -\frac{1}{4}\cos(2t)\right\rangle + \left\langle 0,1,1/4\right\rangle \\ &= \left\langle \frac{1}{3}t^3+t,\, -\sin(t)+t+1,\, -\frac{1}{4}\cos(2t)+1/4\right\rangle.\end{aligned}$$

Problem 5. The position function of a moving particle is given by $\mathbf{r}(t) = \left\langle t^2, 5t, t^2-16t\right\rangle$. The unit of displacement is in meters (m) and the unit of time is in seconds (s). When is the speed a minimum?

HINT: The speed of the particle will give you a 1-variable function in terms of t. Use a Calculus 1 technique to minimize this function.

$\mathbf{r}(t) = \langle t^2, 5t, t^2-16t\rangle \quad\Rightarrow\quad \mathbf{v}(t) = \langle 2t, 5, 2t-16\rangle$, $|\mathbf{v}(t)| = \sqrt{4t^2+25+4t^2-64t+256} = \sqrt{8t^2-64t+281}$

and $\dfrac{d}{dt}|\mathbf{v}(t)| = \frac{1}{2}(8t^2-64t+281)^{-1/2}(16t-64)$. This is zero if and only if the numerator is zero, that is,

$16t-64=0$ or $t=4$. Since $\dfrac{d}{dt}|\mathbf{v}(t)| < 0$ for $t<4$ and $\dfrac{d}{dt}|\mathbf{v}(t)| > 0$ for $t>4$, the minimum speed of $\sqrt{153}$ m/s is attained at $t=4$ units of time.