Section 14.1: Functions of Several Variables

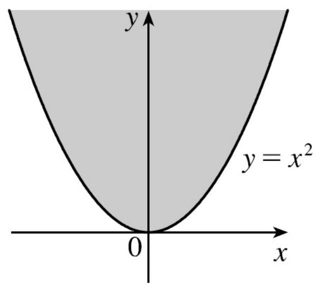
Problem 1. Let $f(x,y) = e^{\sqrt{y-x^2}}$.
(a) Evaluate $f(-2,5)$. (b) Find and sketch the domain of f.
(c) Find the range of f.

(a) We have $h(-2,5) = e^{\sqrt{5-(-2)^2}} = e^{\sqrt{1}} = e$.

(b) Note that $\operatorname{dom}\left(\sqrt{y-x^2}\right) = \{(x,y)|y - x^2 \geq 0\} = \{(x,y)|y \geq x^2\}$. Recall that $\operatorname{dom}(e^t) = \mathbb{R}$, so the function $h(x,y) = e^{\sqrt{y-x^2}}$ can take on any x and y values as long as they are in $\operatorname{dom}(\sqrt{y-x^2})$. That is,

$$\operatorname{dom}(f) = \{(x,y)|y \geq x^2\}.$$

This is precisely the set of all points that lie on and above the parabola $y = x^2$.



(c) Since $\sqrt{t} \geq 0$ for any real number t, $\sqrt{y-x^2} \geq 0$ for any x and y. Then

$$f(x,y) = e^{\sqrt{y-x^2}} \geq e^0 = 1.$$

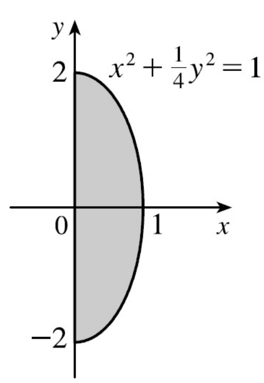
Then $\operatorname{range}(f) = [1, \infty)$.

Problem 2. Find and sketch the domain of
(a) $g(x,y) = \sqrt{x} + \sqrt{4 - 4x^2 - y^2}$ (b) $h(x,y) = \dfrac{\sqrt{xy}}{x+1}$

(a) Note that

$$\operatorname{dom}(g) = \{(x,y) \mid x \geq, 4 - 4x^2 - y^2 \geq 0\} = \left\{(x,y) \mid x \geq 0,\, 1 \geq x^2 + \frac{y^2}{2^2}\right\}.$$

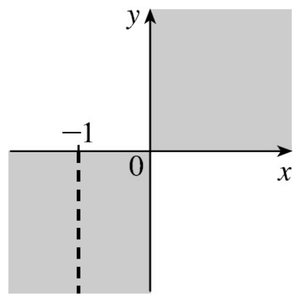
That is, the domain of g consists of all points on or within the right half of a solid ellipse centered at the origin with horizontal radius 1 and vertical radius 2.



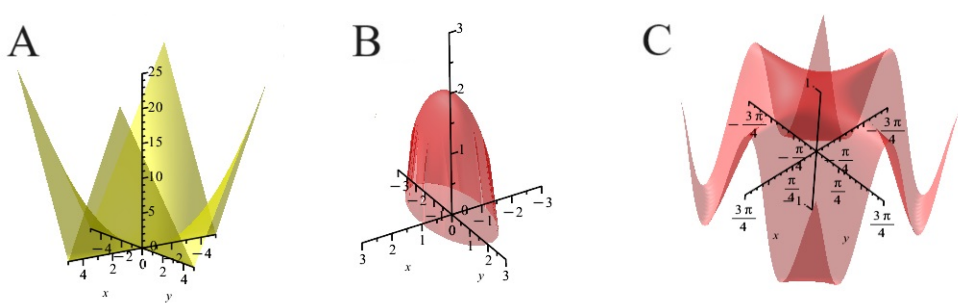
(b) Note that

$$\operatorname{dom}(h) = \{(x,y)\,|\,x \neq -1,\, xy \geq 0\} = \{(x,y)\,|\,x \neq -1,\, x \geq 0, y \geq 0 \text{ or } x \leq 0, y \leq 0\}.$$

That is, the domain of h consists of all points in the first and third quadrants of the xy-plane and exclude all points on the vertical line $x = -1$.



Problem 3. Match each 2-variable function below with their graph.
HINT: Find trace curves with planes that are **not** coordinate planes for (a) and (b).
(a) $f(x,y) = \cos(xy)$ ______ (b) $g(x,y) = |xy|$ ______
(c) $h(x,y) = \sqrt{4 - 4x^2 - y^2}$ ______



(a) The trace of f with the planes $y = 1$ and $x = 1$ are $z = \cos(x)$ and $z = \cos(y)$, respectively. The graph which matches these traces is C.

(b) The trace of f with the planes $y = 1$ and $x = 1$ are $z = |x|$ and $z = |y|$, respectively. The graph which matches these traces is A.

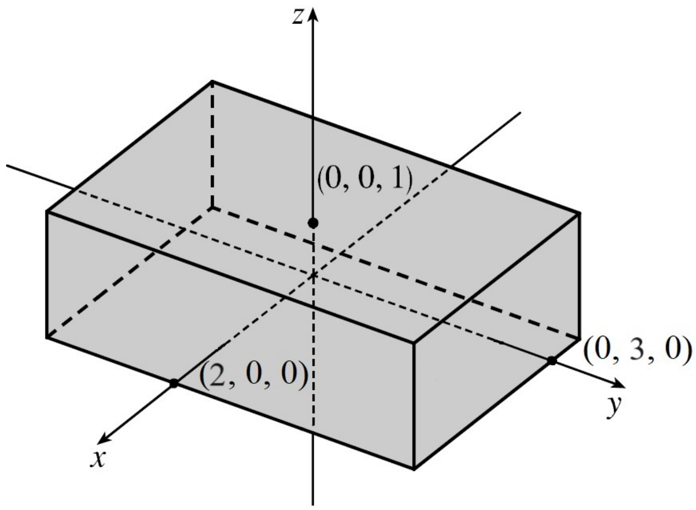
(c) The trace with the plane $x = 0$ is $z = \sqrt{4 - 4x^2}$, which is the upper half of an ellipse of horizontal radius 1 and vertical radius 2 on the xz-plane. The trace with the plane $y = 0$ is $z = \sqrt{4 - y^2}$, which is the upper half of a circle of radius 2 centered at the origin. The trace with the plane $z = 0$ is $4 = 4x^2 + y^2$, or equivalently, $1 = x^2 + y^2/2^2$, which is an ellipse with horizontal radius 1 and vertical radius 2 on the xy-plane. The graph which matches these traces is B.

Problem 4. Find and sketch the domain of
$f(x,y,z) = \sqrt{4 - x^2} + \sqrt{9 - y^2} + \sqrt{1 - z^2}$.

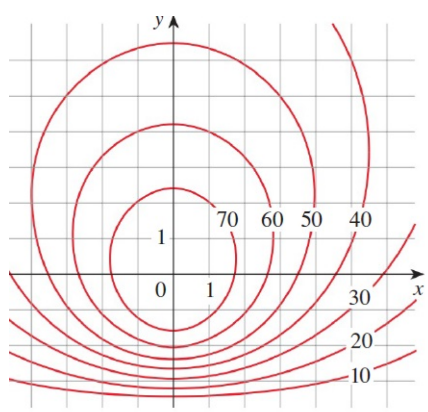
Note that f is a 3-variable function. Then its domain lines in 3D-space. We have

$$\operatorname{dom}(f) = \left\{(x,y,z)\,|\,4 - x^2 \geq 0,\, 9 - y^2 \geq 0,\, 1 - z^2 \geq 0\right\} = \{(x,y,z)\,|\,-2 \leq x \leq 2,\, -3 \leq y \leq 3,\, -1 \leq z \leq 1\}.$$

That is, the domain of f is the set of all points (x, y, z) that lie on or inside the box with dimensions 2 (in the x-direction), 3 (in the y-direction), and 1 (in the z-direction).



Problem 5. A contour map for a function f is shown. Use it to estimate the values of $f(-3, 3)$ and $f(3, -2)$. What can you say about the shape of the graph?

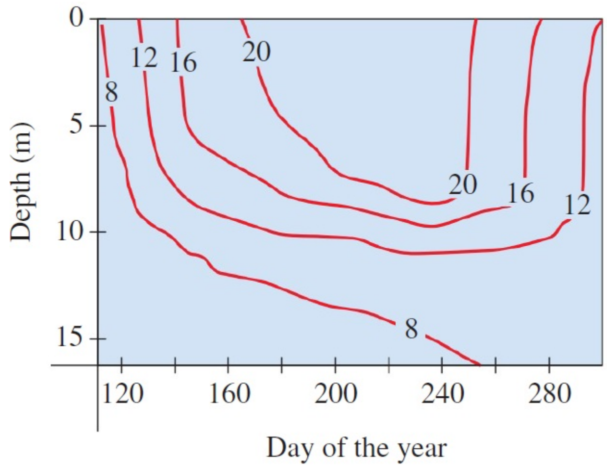


The point $(-3, 3)$ lies between the level curves with z-values 50 and 60. Since the point is a little closer to the level curve with $z = 60$, we estimate that $f(-3, 3) \approx 56$.

The point $(3, -2)$ appears to be just about halfway between the level curves with z-values 30 and 40, so we estimate that $f(3, -2) \approx 35$.

The contour map shows us that the graph rises as we approach the origin, gradually from above, and steeply from below.

Problem 6. Level curves (isothermals) are shown for the typical water temperature (in °C) in Long Lake (Minnesota) as a function of depth and time of year. Estimate the temperature in the lake on June 9 (day 160) at a depth of 10 m and on June 29 (day 180) at a depth of 5 m.



The point $(160, 10)$, corresponding to day 160 and a depth of 10 m, lies between the isothermals with temperature values of 8 and $12°C$. Since the point appears to be located about three-fourths the distance from the $8°C$ isothermal to the $12°C$ isothermal, we estimate the temperature at that point to be approximately $11°C$. The point $(180, 5)$ lies between the 16 and $20°C$ isothermals, very close to the $20°C$ level curve, so we estimate the temperature there to be about $19.5°C$.